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Persistence of Hamiltonian relative periodic orbits

Claudia Wulff[1]

Department of Mathematics and Statistics, University of Surrey, Guildford GU27XH, UK

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Abstract

We prove a persistence result for Hamiltonian relative periodic orbits with generic drift–momentum pairs in the case of non-compact non-free group actions. Our starting point is a relative periodic orbit which is non-degenerate modulo isotropy. We show that the analysis of the persistence problem involves the study of a singular algebraic variety, the space of drift–momentum pairs, which is determined solely by the symmetry group of the problem. We apply our results to relative periodic solutions of deformable bodies in fluids.

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1. Introduction

Relative periodic orbits are periodic solutions of a flow induced by an equivariant vector field on a space of group orbits. In applications they typically appear as oscillations of a system which are periodic when viewed in some rotating or translating frame. They therefore generalize relative equilibria, for which the 'shape' of the system remains constant in an appropriate frame. Relative periodic orbits are ubiquitous in Hamiltonian systems with symmetry. For example, generalizations of the Weinstein–Moser theorem show that they are typically present near stable relative equilibria [22,28] and can therefore be found in virtually any physical application with a continuous symmetry group. Specific examples for which relative periodic orbits have been discussed or could be found by applying

[1] On leave from Freie Universität Berlin, Fachbereich Mathematik und Informatik, Arnimallee 2-6, 14195 Berlin, Germany.
E-mail address: wulff@math.fu-berlin.de (C. Wulff).

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the Weinstein–Moser theorem to stable relative equilibria include rigid bodies [1,21,24], deformable bodies [23,41], molecules [16,18,34] and point vortices [27,31].

In contrast to the situation for general systems [40] so far there are only very partial results on local bifurcation of relative periodic orbits of Hamiltonian systems. These are described below. This is due to the fact that the conservation of momenta and symplectic structure changes the generic behaviour dramatically and has to be taken into account. Most persistence results for Hamiltonian relative periodic orbits which can be found in the literature require compact symmetry groups and so do not apply when there are translational symmetries present. This is frequently the case in applications, e.g. in the case of translating bodies in fluids [21,41] and vortices [31]. So persistence and bifurcations of Hamiltonian relative periodic orbits are still a long way from being well understood, especially in the presence of non-compact symmetry groups.

In a Hamiltonian system without symmetry a periodic orbit is typically a non-degenerate fixed point of the Poincaré map inside its energy level, which implies that periodic orbits appear as one-parameter families parameterized by energy [2,25]. In the case of continuous symmetries with corresponding conserved momenta we expect families of relative periodic orbits to be parameterized by energy and conserved momenta. But a given relative periodic orbit, even if it satisfies a non-degeneracy condition, may not persist to every momentum value nearby, and the persistence problem for relative periodic orbits involves studying to which nearby momentum values a non-degenerate relative periodic orbit persists.

Under a non-degeneracy assumption the following persistence results for Hamiltonian relative periodic orbits have been obtained in the previous work. Montaldi [26] studied persistence to nearby energy–momentum levels of Hamiltonian relative periodic orbits in the case of free actions of compact symmetry groups. Ortega and Ratiu [29] proved a persistence result for non-free group actions by applying [26] to the fixed point space of the isotropy subgroup of the relative periodic orbit. These persistence results apply topological methods which use the compactness of coadjoint group orbits and therefore do not apply when the symmetry group is genuinely non-compact. The paper [41] presents a local description of Hamiltonian vector fields near relative periodic orbits in the case of algebraic symmetry groups. This description, which we summarize in Section 2, is used in [41] to deduce results on persistence for non-degenerate relative periodic orbits of non-compact group actions with regular momentum modulo isotropy. The local description of Wulff and Roberts [41] is also used heavily in this paper.

If the non-degeneracy condition is dispensed with then the bifurcation results which are available mainly deal with fixed points of equivariant symplectic maps [2,5,9,10] which can be applied to the Poincaré map of a periodic orbit inside its energy level, thereby giving bifurcation results for periodic orbits of Hamiltonian systems, see also [20].

The results of this paper are inspired by work on persistence of relative equilibria. For free compact group actions, Patrick [30] proved that there is a manifold of relative equilibria close to a given non-degenerate relative equilibrium if the velocity–momentum pair of the relative equilibrium is regular. If the momentum of the relative equilibrium is regular, then its velocity–momentum pair is also regular, but the latter condition is weaker. For compact symmetry groups G, Patrick and Roberts [32] show that under a generic transversality assumption the analysis of the persistence problem of relative equilibria with general velocity–momentum pairs reduces to the study of a singular algebraic variety, the *space* *of velocity–momentum pairs* $(\mathbf{g} \oplus \mathbf{g}^*)^c$, where $\mathbf{g}$ is the Lie algebra of G and $\mathbf{g}^*$ its dual. This variety is determined solely by the symmetry of the system and so is independent of the given Hamiltonian vector field. A generalization to generic velocity–momentum pairs modulo isotropy and non-compact symmetries is presented in [39].

In this paper we extend the persistence results of Patrick [30] and Wulff [39] on non-degenerate relative equilibria with velocity–momentum pairs which are regular modulo isotropy to relative periodic orbits. We will introduce the *space of drift–momentum pairs* $(G \times \mathbf{g}^*)^c$ which takes the role of the space of velocity–momentum pairs of relative equilibria. Our results are new even for compact group actions, but, as in [40,41], we also deal with non-compact symmetry groups which are algebraic. These are groups defined by polynomial equations and include compact, Euclidean and the classical Lie groups, so this assumption is usually satisfied in applications. We require the relative periodic orbit to be non-degenerate modulo isotropy. This means that it is non-degenerate on the fixed point space of the isotropy of the relative periodic orbit, see Section 2.3. This assumption is quite common in the literature as mentioned above. Our main theorem, Theorem 4.2, treats persistence of non-degenerate relative periodic orbits with regular drift–momentum pairs modulo isotropy to relative periodic orbits with the same spatio-temporal symmetry. In addition, in Theorem 4.7, we give a result on persistence to relative periodic solutions with smaller spatio-temporal symmetry.

The paper is organized as follows. In Section 2 we introduce the setting that we work in and recall the bundle construction near Hamiltonian relative periodic orbits of Wulff and Roberts [41]. In Section 3 we investigate the local structure of the variety $(G \times \mathbf{g}^*)^c$ near regular drift–momentum pairs for general Lie groups. In Section 4 we present our main results, the persistence theorems mentioned above. In Section 5 we apply our methods to oscillations of a deformable body in an ideal fluid.

2. Hamiltonian relative periodic orbits

In this section we describe the setting that we work in and summarize the results of Wulff and Roberts [41] on the bundle structure and differential equations near Hamiltonian relative periodic orbits which we use in the following sections.

We consider a Hamiltonian system

$$\dot{x} = f_{\mathrm{H}}(x) \tag{2.1}$$

on a finite-dimensional symplectic manifold $\mathcal{M}$ with symplectic form $\omega(\cdot, \cdot)$:

$$\omega_x(f_{\mathrm{H}}(x), v) = \mathrm{D}H(x)v, \quad x \in \mathcal{M}, v \in \mathcal{T}_x\mathcal{M}.$$

We assume that a finite-dimensional Lie group G acts on $\mathcal{M}$ properly and symplectically and that the Hamiltonian H is G-invariant. This implies that (2.1) is G-equivariant, i.e.

$$f_{\mathrm{H}}(gx) = gf_{\mathrm{H}}(x), \quad x \in \mathcal{M}, g \in G.$$

So whenever $x(t)$ is a solution of (2.1) then so is $gx(t)$. We call the elements of G the *symmetries* of (2.1). Let $\mathbf{g}$ denote the Lie algebra of G. By Noether's theorem locally there is a conserved quantity $\mathbf{J}_\xi$ for each $\xi \in \mathbf{g}$ such that $\mathbf{J}_\xi$ is the Hamiltonian for the symplectic flow $x \to \exp(\xi t)x$ [1,24]. Moreover, $\mathbf{J}_\xi$ is linear in ξ, so that $\mathbf{J}$ maps to the dual $\mathbf{g}^*$ of the Lie algebra $\mathbf{g}$ of G. Let Ad_g, $g \in G$, denote the adjoint action of G on $\mathbf{g}$: $\mathrm{Ad}_g\,\xi = g\xi g^{-1}$, $\xi \in \mathbf{g}$, $g \in G$, and consider the coadjoint action of G on $\mathbf{g}^*$ given by

$$g\mu = (\mathrm{Ad}_g^*)^{-1}\mu, \quad g \in G. \tag{2.2}$$

We assume throughout the paper that $\mathbf{J}$ is defined on the whole of $\mathcal{M}$ and is G-equivariant with respect to the G-action on $\mathcal{M}$ and the coadjoint action on $\mathbf{g}^*$. For symmetry reduction in the case of only locally defined momentum maps or momentum maps which are not equivariant with respect to the coadjoint action on $\mathbf{g}^*$, see [37].

A point $p \in \mathcal{M}$ lies on a relative periodic orbit if there exists $t > 0$ such that $\Phi_t(p) \in Gp$. The infimum T of such t is called the *relative period* of the relative periodic orbit and the element $\sigma \in G$ such that $\Phi_{\mathrm{T}}(p) = \sigma p$ is called a *phase-shift symmetry*, *reconstruction phase* or *drift symmetry* of the relative periodic orbit. The relative periodic orbit $\mathcal{P}$ itself is given by

$$\mathcal{P} = \{g\Phi_\theta(p), g \in G, \theta \in \mathbb{R}\}.$$

We assume that $T > 0$ so that $\mathcal{P}$ is a proper relative periodic orbit (i.e. not a relative equilibrium). We always reparameterize time such that $T = 1$ and assume, without loss of generality, that $H(p) = 0$. The *spatio-temporal symmetry group* Σ of the relative periodic orbit $\mathcal{P}$ with respect to p is the set of all elements g of G for which there exists $\theta(g) \in \mathbb{R}$ such that $\Phi_{\theta(g)}(p) = gp$. Its elements are called spatio-temporal symmetries of $\mathcal{P}$, and it contains the isotropy subgroup $G_p = \{g \in G, gp = p\}$ of $p \in \mathcal{P}$. In the whole article we assume, as in [41], that G is an algebraic Lie group, i.e. defined by algebraic equations.

In this section we also assume that the isotropy G_p of $p \in \mathcal{P}$ is *finite*. This simplifies the bundle structure near relative periodic orbits and the form of Hamilton's equations in these coordinates considerably.

Remark 2.1. Even if the isotropy subgroup G_p of the relative periodic orbit $\mathcal{P}$ ($p = \sigma^{-1}\Phi_1(p) \in \mathcal{P}$) is not finite on $\mathcal{M}$ it is still finite on the flow-invariant symplectic manifold $\hat{\mathcal{M}} := \mathrm{Fix}_{\mathcal{M}}(\hat{G}_p)$, the fixed point space of $\hat{G}_p$ in $\mathcal{M}$, where $\hat{G}_p \subset G_p$ is a suitable subgroup of the isotropy subgroup G_p of p. The symmetry group acting on $\hat{\mathcal{M}}$ is $L := N(\hat{G}_p)/\hat{G}_p$. Here $N(\hat{G})$ denotes the normalizer of a subgroup $\hat{G}$ of G.

We can always choose $\hat{G}_p$ such that L_p becomes finite by setting $\hat{G}_p = G_p$. Then the action of L near p is *locally free*, i.e. $L_p = \{\mathrm{id}\}$. We call subgroups $\hat{G}_p$ of G_p with L_p is finite *regular* subgroups of G_p, see Definition 4.3 and [39].

We will use this idea to apply persistence results for relative periodic orbits with finite isotropy to the manifold $\hat{\mathcal{M}}$ and in this way get persistence to relative periodic orbits with isotropy containing $\hat{G}_p$ for G_p arbitrary whenever $\hat{G}_p$ is a regular subgroup of G_p. This trick is quite common in the literature, see, e.g. [29,34,39,41]. For more details we refer to Section 4.2.

We end this remark by showing how the drift symmetry σ and the momentum $\mu = \mathbf{J}(p)$ of a relative periodic orbit $\mathcal{P}$ containing $p = \sigma^{-1}\Phi_1(p)$ can be identified with elements

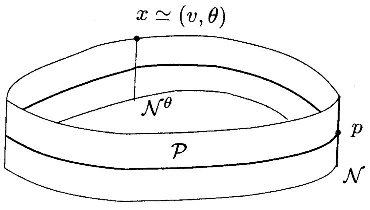

Fig. 1. Coordinates near a periodic orbit $\mathcal{P}$ where $G = \{\mathrm{id}\}$. Here $\mathcal{N}$ and $\mathcal{N}^\theta$ are Poincaré sections to $\mathcal{P}$ at p and $\Phi_\theta(p)$, respectively.

of L and $\mathbf{l}^*$, respectively. Here $\mathbf{l}$ denotes the Lie algebra of L. It is easy to check that the drift symmetry $\sigma \in G$ of the relative periodic orbit lies in $N(G_p)$ [40] and it is only determined modulo G_p, so that it can be identified with $\sigma_\mathrm{L} \in L$. Embedding $\mathbf{l}$ into $\mathbf{g}$ as $\mathbf{l} \simeq \mathrm{Fix}_\mathbf{g}(G_p) \cap \mathbf{g}_p^\perp$ for a G_p-invariant inner product on $\mathbf{g}$ we can identify μ with an element $\mu_\mathrm{L} \in \mathbf{l}^*$ given by $\mu_\mathrm{L} = \mu|_\mathbf{l}$.

2.1. Coordinates near Hamiltonian relative periodic orbits

Let p lie on a relative periodic orbit $\mathcal{P}$ of (2.1) with relative period 1, so $\Phi_1(p) = \sigma p$ for some $\sigma \in G$. Let G_μ be the momentum isotropy of $\mu = \mathbf{J}(p)$ with respect to the coadjoint group action (2.2). For a set $\hat{G} \subset G$ let $\mathbf{z}(\hat{G})$ be the Lie algebra of the centralizer $Z(\hat{G})$ of $\hat{G}$ in G. By [42, Lemma 2.1] any relative periodic orbit of an algebraic group action becomes periodic in a moving frame which respects the isotropy and preserves the momentum of the relative periodic orbit. To be more precise, we can find $\xi \in \mathbf{g}_\mu$, $\alpha \in G_\mu$ such that

$$\sigma = \alpha \exp(\xi), \qquad \mathrm{Ad}_\alpha \xi = \xi, \quad \xi \in \mathbf{z}(G_p), \;\; \exists n \in \mathbb{N}, \;\; \alpha^n = \mathrm{id}. \tag{2.3}$$

So p lies on a periodic orbit with period n when viewed in a frame moving with velocity ξ. Let Σ_n denote the compact group generated by α and G_p. The group Σ_n is the spatio-temporal symmetry group of the relative periodic orbit $\mathcal{P}$ in a frame moving with velocity ξ (and also the spatio-temporal symmetry of the corresponding periodic orbit of the symmetry reduced system, as we will see in Section 2.2). We have $\Sigma_n / G_p = \mathbb{Z}_n^\alpha$.

The tangent space $\mathcal{T}_p\mathcal{M}$ at $p \in \mathcal{P}$ decomposes as $\mathcal{T}_p\mathcal{M} = \mathcal{T} \oplus \mathcal{N}$, where $\mathcal{N}$ is a G_p-invariant Poincaré section to $\mathcal{P}$ at p, i.e. a G_p-invariant complement to $\mathcal{T} := \mathcal{T}_p\mathcal{P}$ in $\mathcal{T}_p\mathcal{M}$. Let $\mathcal{U}$ be a G-invariant neighbourhood of $\mathcal{P}$ in $\mathcal{M}$. Then it is shown in [40,41], see also Figs. 1 and 2, that we can write every $x \in \mathcal{U}$ as $x = (g, v, \theta)$ where $g \in G$, $v \in \mathcal{N}$ lies in the Poincaré section transverse to $\mathcal{P}$ at p and $\theta \in \mathbb{R}$ is the phase of the relative periodic orbit and that these coordinates are unique modulo Σ_n, i.e.,

$$\mathcal{U} \equiv (G \times \mathbb{R}/n\mathbb{Z} \times \mathcal{N})/\Sigma_n. \tag{2.4}$$

The quotient by Σ_n is with respect to the following action of Σ_n on $G \times \mathbb{R}/n\mathbb{Z} \times \mathcal{N}$:

$$(g_p, \alpha^i)(g, \theta, v) = (g\alpha^{-i}g_p^{-1}, \theta + i, g_p Q_\mathcal{N}^i v) \quad \forall g_p \in G_p, \;\; i \in \mathbb{Z}_n, \tag{2.5}$$

where $Q_\mathcal{N} \in \mathrm{O}(\mathcal{N})$ has order n.

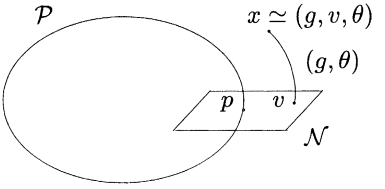

Fig. 2. Coordinates near a relative periodic orbit $\mathcal{P}$, where $G \neq \{\mathrm{id}\}$. Here $\mathcal{N}$ is a Poincaré section to $\mathcal{P}$ at p.

The symplectic structure of the tangent space decomposition $\mathcal{T}_p\mathcal{M} = \mathcal{T} \oplus \mathcal{N}$ and the bundle (2.4) is described in the following theorem.

Theorem 2.2 ([42, Theorem 3.1]). *Let $p = \sigma^{-1}\Phi_1(p) \in \mathcal{P}$ lie on a relative periodic orbit of (2.1) and let G_p be finite. Then there is a choice of Poincaré section $\mathcal{N}$ such that the following hold true*:

(a) *The spaces $\mathcal{T}$ and $\mathcal{N}$ further decompose into*

$$\mathcal{T} = \mathcal{T}_0 \oplus \mathcal{T}_1 \oplus \mathcal{T}_2, \qquad \mathcal{N} = \mathcal{N}_0 \oplus \mathcal{N}_1 \oplus \mathcal{N}_2,$$

where

$$\mathcal{T}_0 = \mathbf{g}_\mu p := \mathcal{T}_p G_\mu p \simeq \mathbf{g}_\mu, \qquad \mathcal{T}_1 = \mathbf{g}p \cap (\mathbf{g}_\mu p)^\perp \simeq \mathbf{g}/\mathbf{g}_\mu,$$
$$\mathcal{T}_2 = \mathrm{span}(f_{\mathrm{H}}(p)) \simeq \mathbb{R},$$

and

$$\mathcal{N}_0 = \ker \mathrm{D}H(p) \cap (\ker \mathrm{D}\mathbf{J}(p))^\perp \cap \mathcal{N} \simeq \mathbf{g}_\mu^*,$$
$$\mathcal{N}_1 = \ker \mathrm{D}H(p) \cap \ker \mathrm{D}\mathbf{J}(p) \cap \mathcal{N},$$
$$\mathcal{N}_2 = (\ker \mathrm{D}H(p))^\perp \cap \ker \mathrm{D}\mathbf{J}(p) \cap \mathcal{N} \simeq \mathbb{R}. \tag{2.6}$$

Orthogonal complements are taken with respect to an appropriate G_p-invariant inner product on $\mathcal{T}_p\mathcal{M}$. The spaces $\mathcal{N}_0$, $\mathcal{N}_1$ and $\mathcal{N}_2$ are all G_p-invariant and $\mathcal{T}_0 \oplus \mathcal{N}_0$, $\mathcal{N}_1$, $\mathcal{T}_1$ and $\mathcal{T}_2 \oplus \mathcal{N}_2$ are symplectic subspaces of $\mathcal{T}_p\mathcal{M}$ such that

$$\omega_p = \omega|_{\mathcal{T}_0 \oplus \mathcal{N}_0} + \omega|_{\mathcal{N}_1} + \omega|_{\mathcal{T}_1} + \omega|_{\mathcal{T}_2 \oplus \mathcal{N}_2}.$$

The symplectic forms on $\mathcal{T}_0 \oplus \mathcal{N}_0$ and $\mathcal{T}_2 \oplus \mathcal{N}_2$ have standard forms in the chosen bundle coordinates.

(b) *Define a Σ_n-invariant symplectic form $\tilde{\omega}$ on $\tilde{\mathcal{U}} := G \times \mathbf{g}_\mu^* \times \mathcal{N}_1 \times \mathcal{T}^*(\mathbb{R}/n\mathbb{Z})$, with $\mathcal{T}^*(\mathbb{R}/n\mathbb{Z}) = \mathbb{R}/n\mathbb{Z} \times \mathcal{N}_2$, by*

$$\tilde{\omega} := \omega_{G \times \mathbf{g}_\mu^*} + \omega_{\mathcal{N}_1} + \omega_{\mathcal{T}^*(\mathbb{R}/n\mathbb{Z})},$$

where $\omega_{G \times \mathbf{g}_\mu^*}$ is the restriction of the standard symplectic form on $\mathcal{T}^*G = G \times \mathbf{g}^*$ to $G \times \mathbf{g}_\mu^*$ with the embedding (2.9), $\omega_{\mathcal{N}_1} := \omega|_{\mathcal{N}_1}$ and $\omega_{\mathcal{T}^*(\mathbb{R}/n\mathbb{Z})}$ is the standard symplectic form on $\mathcal{T}^*(\mathbb{R}/n\mathbb{Z})$. Then the G-reduced phase space

$$\mathcal{U}/G \equiv (\mathbb{R}/n\mathbb{Z} \times \mathcal{N})/\Sigma_n$$

is a Poisson space with respect to the restricted Poisson bracket, and the identification (2.4) *is a symplectomorphism. Moreover, the actions of* G_p *and* $Q_{\mathcal{N}}$ *on* $\mathcal{N}$ *have the forms*:

$$g_p(\nu, w, E) = ((\mathrm{Ad}^*_{g_p})^{-1}\nu, g_p w, E) \quad \forall g_p \in G_p, \tag{2.7}$$

and

$$Q_{\mathcal{N}}(\nu, w, E) = (Q_0\nu, Q_1 w, E) \quad \text{with} \ \ Q_0 = (\mathrm{Ad}^*_\alpha)^{-1}. \tag{2.8}$$

The linear map $Q_1 : \mathcal{N}_1 \to \mathcal{N}_1$ *is orthogonal with respect to the restricted* G_p*-invariant inner product on* $\mathcal{N}_1$, *has order* n *and is symplectic with respect to the restricted* G_p*-semi-invariant symplectic form* $\omega_{\mathcal{N}_1}$.

The space $\mathcal{N}_0 \simeq \mathbf{g}^*_\mu$ can be interpreted as the space of momenta in body coordinates. It is a section transverse to the group orbit $G\mu \subseteq \mathbf{g}^*$ at μ, as we describe in more detail in Lemma 3.7. To embed $\mathbf{g}^*_\mu$ into $\mathbf{g}^*$ we choose a Σ_n-invariant complement $\mathbf{n}_\mu$ to $\mathbf{g}_\mu$ in $\mathbf{g}$. Then we can identify

$$\mathbf{g}^*_\mu \simeq \mathrm{ann}(\mathbf{n}_\mu), \qquad (\mathbf{g}/\mathbf{g}_\mu)^* \simeq \mathrm{ann}(\mathbf{g}_\mu) = \mathcal{T}_\mu G\mu, \tag{2.9}$$

where $\mathrm{ann}(\mathbf{n}_\mu)$ and $\mathrm{ann}(\mathbf{g}_\mu)$ denote the annihilators of $\mathbf{n}_\mu$ and $\mathbf{g}_\mu$ in $\mathbf{g}^*$. Moreover, since G_p is finite the momentum level set $\mathbf{J}^{-1}(\mu) \subseteq \mathcal{M}$ is a manifold near p so that $\mathcal{N}_1$ can be interpreted as a Poincaré section transverse to the relative periodic orbit $\mathcal{P}_\mu = \mathcal{P} \cap \mathbf{J}^{-1}(\mu)$ inside the energy–momentum level set $\mathbf{J}^{-1}(\mu) \cap H^{-1}(p)$. Physically, it typically describes shape vibrations. The parameter $E \in \mathcal{N}_2$ parameterizes the energy level sets. The variable $\theta \in \mathbb{R}/n\mathbb{Z}$ is the phase of the relative periodic orbit [41].

Remark 2.3. As shown in [41], the tangent space decomposition of Theorem 2.2(a) remains true for continuous isotropy groups G_p if we now identify $\mathcal{T}_0 \simeq \mathbf{g}_\mu/\mathbf{g}_p$ and $\mathcal{N}_0 \simeq (\mathbf{g}_\mu/\mathbf{g}_p)^*$. We will need this for the study of symmetry breaking bifurcations in Section 4.2. The modifications of Theorem 2.2(b) which are necessary in the case of continuous isotropy groups are more complicated, see [41], and not needed in this paper.

2.2. *Hamiltonian systems near relative periodic orbits*

In this section we present the differential equations near a relative periodic orbit $\mathcal{P}$ expressed in the bundle coordinates that we introduced in the previous section. As we will see they decompose the dynamics near $\mathcal{P}$ into a periodically forced motion inside a Poincaré section which drives the drift dynamics on the group.

To describe these differential equations we first introduce some more notation. We denote the identity component of a subgroup $\tilde{G}$ of G by $\tilde{G}^{\mathrm{id}}$ and say that an element $\mu \in \mathbf{g}^*$ *split* [14,36,41], σ-*split* for some $\sigma \in G_\mu$, or *strongly* split, if the complement $\mathbf{n}_\mu$ to $\mathbf{g}_\mu$ in $\mathbf{g}$ can be chosen to be invariant under G^{id}_μ, G^{id}_μ and Ad^*_σ or G_μ, respectively. This is always the case if G is compact, but in general is not satisfied if G is non-compact. An example is provided by the Euclidean group $G = \mathrm{SE}(3)$ of motions in three-space, see [36].

Let $P_{\mathrm{ann}(\mathbf{g}_\mu)}$ be the projection from $\mathbf{g}^*$ to $\mathrm{ann}(\mathbf{g}_\mu)$ with kernel $\mathrm{ann}(\mathbf{n}_\mu) \simeq \mathbf{g}_\mu^*$. Then for any $\xi \in \mathbf{g}_\mu$ and any small $\nu \in \mathrm{ann}(\mathbf{n}_\mu)$ there is a unique $\eta \in \mathbf{n}_\mu$ such that

$$P_{\mathrm{ann}(\mathbf{g}_\mu)}(\mathrm{ad}^*_{\xi+\eta}(\mu+\nu)) = 0, \tag{2.10}$$

as is shown in [37, Proposition 2.5]. Moreover, $\eta = \eta_\mu(\nu)\xi$ is linear in ξ. If μ is split $\mathbf{n}_\mu$ can be chosen G_μ^{id}-invariant. Then $\mathrm{ann}(\mathbf{n}_\mu) \simeq \mathbf{g}_\mu^*$ is G_μ^{id}-invariant which implies that $\eta \equiv 0$. For every small $\nu \in \mathrm{ann}(\mathbf{n}_\mu)$ define the linear map $j_\mu(\nu)$ from $\mathbf{g}_\mu$ to $\mathbf{g}$ as $j_\mu(\nu) = \mathrm{id} + \eta_\mu(\nu)$. We denote by Ad^μ and ad^μ the adjoint action of G_μ and $\mathbf{g}_\mu$ on $\mathbf{g}_\mu$.

Let $\hat{h} = \hat{h}(\theta, \nu, w, E)$ denote the lift of the G-invariant Hamiltonian H back to the space $G \times \mathbb{R}/n\mathbb{Z} \times (\mathcal{N}_0 \oplus \mathcal{N}_1 \oplus \mathcal{N}_2)$ under the map given by Theorem 2.2. The function $\hat{h}$ is Σ_n-invariant:

$$\hat{h}(\theta, (\mathrm{Ad}^*_{g_p})^{-1}\nu, g_p w, E) = \hat{h}(\theta, \nu, w, E) \quad \forall g_p \in G_p,$$

and

$$\hat{h}(\theta+1, (\mathrm{Ad}^*_\alpha)^{-1}\nu, Q_1 w, E) = \hat{h}(\theta, \nu, w, E).$$

In particular, $\hat{h}$ is periodic in θ with period n. We are now ready to formulate the Hamiltonian system (2.1) in the bundle coordinates (2.4).

Theorem 2.4 ([42, Theorems 3.3 and 3.4]). *Let $p = \sigma^{-1}\Phi_1(p) \in \mathcal{P}$ lie on a relative periodic orbit $\mathcal{P}$ with finite isotropy subgroup G_p and let $\sigma = \alpha\exp(\xi)$ as in (2.3). Assume that time is parameterized so that the phase dynamics near the relative periodic orbit is given by $\dot{\theta} \equiv 1$. Then the Hamiltonian $\hat{h}$ in bundle coordinates is of the form*

$$\hat{h}(\theta, \nu, w, E) = h(\theta, \nu, w) + E \tag{2.11}$$

for some Σ_n-invariant function h on $\mathbb{R}/n\mathbb{Z} \times (\mathcal{N}_0 \oplus \mathcal{N}_1)$. We have $\mathrm{D}_{(\nu,w)}h(\theta, 0, 0) = (\xi, 0)$ and the differential equations for the motion in bundle coordinates

$$(g, \theta, \nu, w, E) \in G \times \mathbb{R}/n\mathbb{Z} \times \mathcal{N}$$

have the form

$$\begin{aligned}
\dot{g} &= g j_\mu(\nu)\,\mathrm{D}_\nu h(\theta, \nu, w),\\
\dot{\nu} &= \mathrm{ad}^*_{j_\mu(\nu)\,\mathrm{D}_\nu h(\theta,\nu,w)}(\nu+\mu) = \mathrm{ad}^{\mu,*}_{\mathrm{D}_\nu h}\,\nu + P_{\mathbf{g}_\mu^*}(\mathrm{ad}^*_{\hat{\eta}_\mu(\theta,\nu,w)}(\nu)),\\
\dot{w} &= J_{\mathcal{N}_1}\,\mathrm{D}_w h(\theta, \nu, w),\\
\dot{E} &= -\mathrm{D}_\theta h(\theta, \nu, w),\\
\dot{\theta} &= 1,
\end{aligned} \tag{2.12}$$

where $\hat{\eta}(\theta, \nu, w) = \eta_\mu(\nu)\,\mathrm{D}_\nu h(\theta, \nu, w)$. Here $\mathrm{ad}^{\mu,*}_\chi$, $\chi \in \mathbf{g}_\mu$, is the dual operator to ad^μ_χ and $P_{\mathbf{g}_\mu^*}$ the projection from $\mathbf{g}^*$ to $\mathrm{ann}(\mathbf{n}_\mu) \simeq \mathbf{g}_\mu^*$ with kernel $\mathrm{ann}(\mathbf{g}_\mu)$.

Note that $\mu + \nu(t)$ is the momentum in a frame moving with velocity $j_\mu(\nu(t))\,\mathrm{D}_\nu h(t, \nu(t), w(t))$, so if $(g(t), \nu(t), w(t))$ is a solution of (2.12) and $g(0) = \mathrm{id}$, then the momentum conservation in bundle coordinates reads

$$g(t)(\mu + \nu(t)) = \mu + \nu(0). \tag{2.13}$$

Eq. (2.12) without the g-equation are called the *symmetry reduced system*. These equations form a Σ_n-invariant periodically driven Poisson system on the Poincaré section $\mathcal{N}$ together with the (trivial) phase dynamics of θ. The relative periodic orbit becomes a periodic orbit of the symmetry reduced system with spatio-temporal symmetry group Σ_n. Any relative periodic orbit $\hat{\mathcal{P}}$ close to $\mathcal{P}$ also becomes a periodic orbit of the symmetry reduced system, and we say that $\mathcal{P}$ and $\hat{\mathcal{P}}$ have the *same reduced spatio-temporal symmetry* if $\hat{\mathcal{P}}$ corresponds to a periodic solution with spatio-temporal symmetry Σ_n for the symmetry-reduced system near $\mathcal{P}$. This does *not* necessarily imply that the spatio-temporal symmetry groups Σ and $\hat{\Sigma}$ of $\mathcal{P}$ and $\hat{\mathcal{P}}$ coincide. For example for $G = \mathrm{SO}(2)$ it may happen that $\mathcal{P}$ is a quasiperiodic solution whereas rotational and relative periods of $\hat{\mathcal{P}}$ may be rationally dependent, so that $\hat{\mathcal{P}}$ is fibered by periodic solutions. Note that $\mathcal{P}$ and $\hat{\mathcal{P}}$ have the same reduced spatio-temporal symmetry if and only if $\mathcal{P}$ and $\hat{\mathcal{P}}$ have the same isotropy and the same relative period (where we use the time-reparameterization of Theorem 2.4).

2.3. Non-degenerate relative periodic orbits

In this section we introduce a non-degeneracy condition for relative periodic orbits which we will use in our persistence results in Section 4. We will see that this non-degeneracy assumption reduces the persistence problem to the study of the structure of a certain variety, the space of drift–momentum pairs.

We are mainly interested in the persistence of a relative periodic orbit $\mathcal{P}$ to nearby relative periodic orbits with the same reduced spatio-temporal symmetry, that is, the same isotropy and the same relative period (with respect to the time parameterization of Theorem 2.4), and restrict to this case in this subsection. But in Section 4 we will also consider persistence to relative periodic orbits with smaller reduced spatio-temporal symmetry group. Since we restrict attention to nearby relative periodic orbits with the same isotropy we can, without loss of generality, assume that G_p is finite, for example by replacing $\mathcal{M}$ with $\hat{\mathcal{M}} = \mathrm{Fix}_{\mathcal{M}}(G_p)$, cf. Remark 2.1.

Before we can define the notion of a non-degenerate relative periodic orbit we need another proposition.

Proposition 2.5 ([42, Proposition 4.3]). *Let $p = \sigma^{-1}\Phi_1(p) \in \mathcal{P}$, $\sigma \in G$, lie on a relative periodic orbit $\mathcal{P}$ with finite isotropy subgroup G_p and let $B = \sigma^{-1}\,\mathrm{D}\Phi_1(p)$. Let time be reparameterized so that $\dot{\theta} \equiv 1$. Then:*

(a) *The map B has the following structure with respect to the decomposition $\mathcal{T}_p\mathcal{M} = \mathcal{T}_pGp \oplus \mathbb{R} \oplus \mathcal{N}$:*

$$B = \begin{pmatrix} \mathrm{Ad}_\sigma^{-1} & 0 & D \\ 0 & 1 & 0 \\ 0 & 0 & B_{\mathcal{N}} \end{pmatrix}. \tag{2.14}$$

(b) *Let $v \to \Phi_{1,0}^{\mathcal{N}}(v)$, $v = (\nu, w, E) \in \mathcal{N}$, be the time 1 map of the periodically forced system on $\mathcal{N}$. Then 0 is a fixed point of the map $Q_{\mathcal{N}}^{-1}\Phi_{1,0}^{\mathcal{N}}$. The block $B_{\mathcal{N}}$ in (2.14) is the linearization of this map, i.e. $B_{\mathcal{N}} = Q_{\mathcal{N}}^{-1}\,\mathrm{D}\Phi_{1,0}^{\mathcal{N}}(0)$, and has the following block structure* *with respect to the decomposition* $\mathcal{N} = \mathcal{N}_0 \oplus \mathcal{N}_1 \oplus \mathcal{N}_2$:

$$B_{\mathcal{N}} = \begin{pmatrix} \mathrm{Ad}_\sigma^{\mu,*} & 0 & 0 \\ B_{10} & B_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We are now ready to define what a non-degenerate relative periodic orbit is.

Definition 2.6. Let $p = \sigma^{-1}\Phi_1(p)$ lie on a relative periodic orbit with finite isotropy subgroup G_p. The relative periodic orbit $\mathcal{P}$ is *non-degenerate modulo isotropy* if B_1 does not have eigenvectors with eigenvalue 1. Here B_1 is the block in $B = \sigma^{-1}\,\mathrm{D}\Phi_1(p)|_{\mathrm{Fix}_{\mathcal{M}}(G_p)}$ defined in Proposition 2.5.

Remark 2.7. In Section 4.2 we adapt the notion of non-degeneracy and the considerations below to symmetry breaking bifurcations.

Let Ψ_{t,t_0} be the time-evolution of the periodically forced differential equations on $\mathcal{N}_0 \oplus \mathcal{N}_1$, that is, inside the energy level set $E = H(p)$ of the Poincaré section $\mathcal{N}$. Let $Q_{\mathcal{N}_0\oplus\mathcal{N}_1} = Q_{\mathcal{N}}|_{\mathcal{N}_0\oplus\mathcal{N}_1}$. Then 0 is a fixed point of $Q^{-1}_{\mathcal{N}_0\oplus\mathcal{N}_1}\Psi_{1,0}$ and relative periodic orbits close to $\mathcal{P}$ in $\mathcal{M}$ correspond to periodic points of $Q^{-1}_{\mathcal{N}_0\oplus\mathcal{N}_1}\Psi_{1,0}$. Write $\Psi_{1,0} = (\Psi^\nu_{1,0}, \Psi^w_{1,0})$. As shown in Proposition 2.5 we have $Q_1^{-1}\,\mathrm{D}\Psi^w_{1,0}(0) = B_1$, so, if $\mathcal{P}$ is non-degenerate, we can solve the equation $Q_1^{-1}\Psi^w_{1,0}(\nu, w) = w$ uniquely for $w(\nu) \in \mathcal{N}_1$ if $\nu \in \mathcal{N}_0$ is small. Therefore the problem of finding relative periodic orbits close to $\mathcal{P}$ in $\mathcal{M}$ with relative period close to 1 and same isotropy as $\mathcal{P}$ is equivalent to finding the solutions of the fixed point equation

$$\Pi(\nu) = \nu, \quad \text{where} \;\; \Pi : \mathrm{Fix}_{\mathcal{N}_0}(G_p) \to \mathcal{N}_0, \qquad \Pi(\nu) = \mathrm{Ad}^*_\alpha\,\Psi^\nu_{1,0}(\nu, w(\nu)). \tag{2.15}$$

Here we used the block structure of $Q_{\mathcal{N}}$ given in (2.8). If ν is any such fixed point of Π then $(\nu, w(\nu))$ is a fixed point of $Q^{-1}_{\mathcal{N}_0\oplus\mathcal{N}_1}\Psi_{1,0}$ with isotropy subgroup G_p. Since $\partial_t h(t, \Psi^\nu_{t,0}, \Psi^w_{t,0}) = \mathrm{D}_t h(t, \Psi^\nu_{t,0}, \Psi^w_{t,0})$ for every trajectory $\Psi_{t,0}(\nu, w)$ of the periodically forced (ν, w)-subsystem of (2.12) and since the Hamiltonian in bundle coordinates $h(\theta, \nu, w)$ is Σ_n-invariant by Theorem 2.4, we get from the E-equation of (2.12) that

$$\begin{aligned} E(1) - E(0) &= \int_0^1 \mathrm{D}_t h(t, \Psi^\nu_{t,0}(\nu, w(\nu)), \Psi^w_{t,0}(\nu, w(\nu)))\,\mathrm{d}t \\ &= h(1, Q_0\nu, Q_1 w(\nu)) - h(0, \nu, w(\nu)) = 0. \end{aligned}$$

Therefore the fixed point $(\nu, w(\nu))$ of $Q^{-1}_{\mathcal{N}_0\oplus\mathcal{N}_1}\Psi_{1,0}$ generates a one-parameter family of fixed points of $Q^{-1}_{\mathcal{N}}\Phi^{\mathcal{N}}_{1,0}$ parameterized by energy E. These fixed points lie on periodic solutions of the periodically forced symmetry reduced system on the Poincaré section $\mathcal{N}$. Using again the bundle equation (2.12) we see that this one-parameter family of periodic solutions on the Poincaré section $\mathcal{N}$ corresponds to a one-parameter family of relative periodic solutions of the original system (2.1). This procedure is quite common [26,29,41] and the same idea is used in the study of persistence of relative equilibria [29,34, 36,39].

To analyse the properties of the map Π from (2.15) we need to describe the symplectic leaves of the space $\mathcal{N}_0 \simeq \mathbf{g}_\mu^*$ which is a Poisson space by Theorem 2.2. Since the Poisson structure on $\mathcal{N}_0$ is in general not the standard Lie–Poisson structure unless μ is split [36], the symplectic leaves are in general not coadjoint orbits. To describe the symplectic leaves we need one more notion. Let $\mu \in \mathbf{g}^*$. As in [39] we define $\tilde{Z}_{\mu,\nu}$, where $\nu \in \mathrm{ann}(\mathbf{n}_\mu) \simeq \mathbf{g}_\mu^*$ is small, by the condition

$$g \in \tilde{Z}_{\mu,\nu} \Leftrightarrow P_{\mathrm{ann}(\mathbf{g}_\mu)}(\mathrm{Ad}^*_{g^{-1}}(\mu+\nu)-\mu) = 0 \tag{2.16}$$

and $Z_{\mu,\nu}$ to be the path connected component of $\tilde{Z}_{\mu,\nu}$ containing the identity. Clearly, $G^{\mathrm{id}}_{\mu+\nu} \subseteq Z_{\mu,\nu}$ for every $\nu \in \mathrm{ann}(\mathbf{n}_\mu)$ and $\mathcal{T}_{\mathrm{id}} Z_{\mu,\nu} = j_\mu(\nu)\mathbf{g}_\mu$.

Lemma 2.8. *Let $p = \sigma^{-1}\Phi_1(p) \in \mathcal{P}$ lie on a relative periodic orbit with finite isotropy subgroup G_p.*

(a) *The symplectic leaves of $\mathcal{N}_0 \simeq \mathbf{g}_\mu^* \simeq \mathrm{ann}(\mathbf{n}_\mu)$ are given by $L(\nu) = Z_{\mu,\nu}(\mu+\nu)-\mu$. If μ is split then $L(\nu) = G^{\mathrm{id}}_\mu \nu$ is a coadjoint orbit.*
(b) *The map Π from (2.15) is Poisson and has the form*

$$\Pi(\nu) = g(\nu)(\mu+\nu)-\mu, \quad g(\nu) \in \tilde{Z}_{\mu,\nu} \tag{2.17}$$

with $g : \mathrm{Fix}_{\mathcal{N}_0}(G_p) \to G$ smooth and $g(0) = \sigma^{-1}$. If μ is strongly split and $\mathbf{n}_\mu$ is chosen to be G_μ-invariant then $g(\nu) \in G_\mu$ so that $\Pi(\nu) = g(\nu)\nu$.

Proof.

(a) The symplectic leaves of $\mathbf{g}^*$ are the coadjoint orbits $G^{\mathrm{id}}(\mu+\nu)$. On $\mu + \mathrm{ann}(\mathbf{n}_\mu)$ they restrict to the leaves $Z_{\mu,\nu}(\mu+\nu)$.
(b) Let $g(t) = \Phi_t^G(\nu, w)$ denote the solution of the g-equation of (2.12) with initial value (id, ν, w). Then the formula (2.13) for the momentum $\nu(t)$ in the comoving frame, and the definition (2.15) of Π imply (2.17) with $g(\nu) = (\Phi_1^G(\nu, w(\nu))\alpha)^{-1}$. The rest of (b) is clear. □

Let

$$\pi : G \times \mathbf{g}^* \to \mathbf{g}^*, \qquad \pi(g,\mu) = g\mu, \tag{2.18}$$

let $P_{\mathbf{g}^*} : G \times \mathbf{g}^* \to \mathbf{g}^*$, $P_{\mathbf{g}^*}(g,\nu) = \nu$, be the projection from $G \times \mathbf{g}^*$ to $\mathbf{g}^*$ and let

$$(G \times \mathbf{g}^*)^{\mathrm{c}} = (\pi - P_{\mathbf{g}^*})^{-1}(0) = \{(g,\mu) \in G \times \mathbf{g}^*, g\mu = \mu\} \tag{2.19}$$

be the set of elements of $G \times \mathbf{g}^*$ which commute in the sense that $g\mu = \mu$, or equivalently $\pi(g,\mu) = \mu$. Since G is an algebraic group this is in general a singular *algebraic variety*. The drift $\sigma \in G$ of a relative periodic orbit $\mathcal{P}$ in $\mathcal{M}$ with momentum μ satisfies $\sigma \in G_\mu$ so that $(\sigma,\mu) \in (G \times \mathbf{g}^*)^{\mathrm{c}}$. We therefore call $(G \times \mathbf{g}^*)^{\mathrm{c}}$ the *space of drift–momentum pairs*. From Lemma 2.8 we see that the map Π describing relative periodic orbits near $\mathcal{P}$ and the map π determining $(G \times \mathbf{g}^*)^{\mathrm{c}}$ are related by

$$\Pi(\nu) = \pi(g(\nu), \mu+\nu) - \mu. \tag{2.20}$$

This is very similar to the situation for the set of non-degenerate relative equilibria which is determined by the space of velocity–momentum pairs [32,39] given by the variety

$$(\mathbf{g} \oplus \mathbf{g}^*)^{\mathrm{c}} = \mathcal{T}_{(\mathrm{id},0)}(G \times \mathbf{g}^*)^{\mathrm{c}}.$$

The following proposition summarizes the relationship between solutions of (2.15) and (2.20), and relative periodic orbits of (2.1).

Proposition 2.9. *Let p lie on a relative periodic orbit $\mathcal{P}$ with finite isotropy subgroup G_p and energy $H(p) = 0$ which is non-degenerate modulo isotropy. Then every relative periodic orbit close to $\mathcal{P}$ with isotropy subgroup G_p and relative period 1 corresponds to a fixed point of Π as defined in (2.15). Let $\nu(\lambda)$, $\lambda \in \mathbb{R}^r$, $\nu(0) = 0$, be an r-dimensional solution manifold of (2.15) and (2.20). Then there is an $(r+1)$-dimensional family of relative periodic orbits $\mathcal{P}(\lambda, E)$ in $\mathcal{M}$ with isotropy subgroup G_p, momentum $\mathbf{J}(p(\lambda, E)) = \mu + \nu(\lambda)$, $p(\lambda, E) \in \mathcal{P}(\lambda, E)$, relative period one (where time has been reparameterized such that $\dot{\theta} \equiv 1$) and drift symmetry $\sigma(\lambda, E)$, with $\sigma(0,0) = \sigma$, $p(0,0) = p$, $\mathcal{P}(0,0) = \mathcal{P}$. Moreover $p(\lambda, E)$ and $\sigma(\lambda, E)$ are parameterized smoothly by $\lambda \in \mathbb{R}^r$ and energy E. The manifold $\mathcal{M}_{\mathrm{RPO}}$ in $\mathcal{M}$ formed by this $(r+1)$-dimensional family of relative periodic orbits has dimension*

$$\dim \mathcal{M}_{\mathrm{RPO}} = \dim G + r + 2.$$

Remark 2.10. The above proposition deals with persistence to relative periodic orbits with the same reduced spatio-temporal symmetry group, but the arguments can easily be adapted to give persistence to relative periodic orbits with smaller reduced spatio-temporal symmetry group. For details, see Section 4.2.

We see that in order to obtain persistence results for non-degenerate relative periodic orbits we need to understand the structure of the space of drift–momentum pairs $(G \times \mathbf{g}^*)^{\mathrm{c}}$. In Section 3 we study the variety $(G \times \mathbf{g}^*)^{\mathrm{c}}$ near regular points. After that, in Section 4, we solve (2.15) under a genericity assumption on the drift–momentum pair. In this way we prove our main result, Theorem 4.2, on persistence of non-degenerate relative periodic orbits with regular drift–momentum pairs.

3. Regular drift–momentum pairs

In this section we investigate the variety $V = (G \times \mathbf{g}^*)^{\mathrm{c}}$ of drift–momentum pairs defined in (2.19). We introduce the notion of a regular drift–momentum pair, describe the variety locally using a section transverse to the group orbit, and give sufficient conditions for a drift–momentum pair to be regular. The results are analogous to results on the space $(\mathbf{g} \oplus \mathbf{g}^*)^{\mathrm{c}}$ of velocity–momentum pairs of relative equilibria of [32,39].

Let, as before, $Z(g)$ be the centralizer of $g \in G$. Define an action of $g \in G$ on $\sigma \in G$ by $g\sigma g^{-1}$ and let $G_\sigma = Z(\sigma)$ be the isotropy group of σ with respect to this action of G. The following concepts generalize the notions of Patrick et al. [30,32,39] to relative periodic orbits.

Definition 3.1.

(a) We call $\mu \in \mathbf{g}^*$ *regular* (or *minimal* [36]) if its isotropy subgroup G_μ for the coadjoint action of G on $\mathbf{g}^*$ has minimal dimension $r_\mu(G)$.
(b) We call $\sigma \in G$ *regular* (or *minimal*) if the dimension $r_\sigma(G)$ of its isotropy G_σ is locally minimal.
(c) A pair $(\sigma, \mu) \in (G \times \mathbf{g}^*)^{\mathrm{c}}$ is called *regular* if $(G \times \mathbf{g}^*)^{\mathrm{c}}$ is a manifold near (σ, μ). In this case let $r = r_{(\sigma,\mu)}(G)$ be such that $(G \times \mathbf{g}^*)^{\mathrm{c}}$ has dimension $\dim_{(\sigma,\mu)}(G \times \mathbf{g}^*)^{\mathrm{c}} = \dim G + r$ near (σ, μ).
(d) Define a G-action on $G \times \mathbf{g}^*$ by

$$g(\sigma, \mu) = (g\sigma g^{-1}, g\mu), \quad \sigma \in G, \ \ \mu \in \mathbf{g}^*, \ \ g \in G,$$

so that $G_{(\sigma,\mu)} = G_\sigma \cap G_\mu$. A drift–momentum pair $(\sigma, \mu) \in (G \times \mathbf{g}^*)^{\mathrm{c}}$ is called *minimal* if $\dim G_{(\sigma,\mu)}$ is locally minimal in $(G \times \mathbf{g}^*)^{\mathrm{c}}$.

Remark 3.2. If (σ, μ) is a minimal drift–momentum pair then σ is minimal in G_μ and the two conditions are equivalent if μ is σ-split as we will see in Proposition 3.11. In this proposition we also show that minimality and regularity of a pair $(\sigma, \mu) \in (G \times \mathbf{g}^*)^{\mathrm{c}}$ are equivalent if μ is σ-split. In particular, this is always true for compact groups, see also Proposition 3.3. In general the relation between minimality and regularity of drift–momentum pairs is a non-trivial problem of algebraic geometry, as for velocity–momentum pairs $(\xi, \mu) \in (\mathbf{g} \oplus \mathbf{g}^*)^{\mathrm{c}}$ of relative equilibria [39], see also Remark 3.6(a) and (b).

3.1. Compact groups

In the case of a compact group G the sets of regular elements $\sigma \in G$, $\mu \in \mathbf{g}^*$ and $(\sigma, \mu) \in (G \times \mathbf{g}^*)^{\mathrm{c}}$ are easily classified. Recall that a *Cartan subgroup* $C(\sigma)$ of a compact group G is a subgroup of G which is generated by a single element $\sigma \in G$ and is not contained properly in another group with this property. Elements $\sigma \in G$ generating a Cartan subgroup are open and dense in G, and if σ_1, σ_2 lie in the same connected component of G then they lie in conjugate Cartan subgroups [7]. A maximal torus T is a Cartan subgroup $T = C(\sigma)$ with $\sigma \in G^{\mathrm{id}}$, and for any $\sigma \in G$ generating a Cartan subgroup $C(\sigma)$ the identity component $C(\sigma)^{\mathrm{id}}$ of $C(\sigma)$ is a maximal torus in $Z(\sigma)$. We have the following proposition.

Proposition 3.3. *In the case of a compact group G a pair $(\sigma, \mu) \in (G \times \mathbf{g}^*)^{\mathrm{c}}$ is regular if and only if it is minimal. Moreover $\mu \in \mathbf{g}^*$ is minimal if and only if $\mathbf{g}_\mu$ is the Lie algebra of a maximal torus in G, and $\sigma \in G$ and $(\sigma, \mu) \in (G \times \mathbf{g}^*)^{\mathrm{c}}$ are minimal if and only if $\mathbf{g}_\sigma$ and $\mathbf{g}_{(\sigma,\mu)}$ are Lie algebras of Cartan subgroups in G. Finally, if (σ, μ) is regular then $r_{(\sigma,\mu)}(G) = \dim \mathbf{g}_{(\sigma,\mu)}$.*

Proof. From Patrick [30] and Wulff [39] we know that $\mu \in \mathbf{g}^*$ is minimal if and only if $\mathbf{g}_\mu$ is the Lie algebra of a maximal torus of G.

As mentioned above every $\sigma \in G$ is contained in a Cartan subgroup C. Then the Lie algebra $\mathbf{c}$ of C is contained in $\mathbf{g}_\sigma$. As a consequence $\mathbf{g}_\sigma$ has minimal dimension if and only if $\mathbf{c} = \mathbf{g}_\sigma$.

Since for compact groups every μ is strongly split it follows from Proposition 3.11(c) that $(\sigma, \mu) \in (G \times \mathbf{g}^*)^c$ is regular if and only if it is minimal. Let $(\sigma, \mu) \in (G \times \mathbf{g}^*)^c$ be arbitrary. Identifying adjoint and coadjoint actions and using that $\sigma\mu = \mu$ we can identify μ with some element of $\mathbf{z}(\sigma)$, and so there is a maximal torus in $Z(\sigma)$ with Lie algebra $\mathbf{t}_\sigma$ such that $\mathbf{t}_\sigma \subseteq \mathbf{g}_{(\sigma,\mu)}$. The Lie algebra $\mathbf{t}_\sigma$ is the Lie algebra of a Cartan subgroup C containing σ. The proof now follows, as before, from the fact that $\mathbf{g}_{(\sigma,\mu)}$ has locally minimal dimension if and only if $\mathbf{g}_{(\sigma,\mu)} = \mathbf{t}_\sigma$. $\square$

3.2. *Regularity and algebraic geometry*

In this section we prove that regular elements $\sigma \in G$, $\mu \in \mathbf{g}^*$ and $(\sigma, \mu) \in (G \times \mathbf{g}^*)^c$ are generic and that $r_\sigma(G)$ is constant on connected components of G (Proposition 3.5). To do this we introduce some elementary real algebraic geometry. An (algebraic) variety $V \subset \mathbb{R}^m$ is an algebraic set, i.e., a subset of $\mathbb{R}^m$ defined by polynomial equations $f_1(x) = \cdots = f_n(x) = 0$, where $x = (x_1, \dots, x_m) \in \mathbb{R}^m$ and $f_i \in \mathbb{R}[x]$, $i = 1, \dots, n$. We write $V = V(f)$. Here $\mathbb{R}[x] = \mathbb{R}[x_1, \dots, x_m]$ denotes the ring of polynomials in m variables. An algebraic variety is called *irreducible* if it is not a union of two different non-empty algebraic varieties [4]. Recall that a set $U \subset V$ is called Zariski-open if $V \setminus U$ is an algebraic variety. So we see that an algebraic variety V is irreducible if any Zariski-open sets U_1, U_2 of V intersect. As in [13] we call a point $p \in V$ *regular* or non-singular if p is a *smooth* point of V, that is, V is a manifold near p, otherwise p is called *singular*. Smooth points of algebraic varieties are open and dense [13, Theorem I.2.4].

We will need the following lemma on algebraic groups in the sequel.

Lemma 3.4 ([15, Theorem II.1.4]). *Let G be algebraic. Then its irreducible components are disjoint and are unions of connected components of G.*

We call $\xi \in \mathbf{g}$ regular if $\mathbf{g}_\xi$ has minimal dimension [30,39] and define $r_\xi(G) = \dim(\mathbf{g}_\xi)$. We are now ready to state the main result of this section.

Proposition 3.5.

(a) *The set of regular elements $\mu \in \mathbf{g}^*$ is open and dense in $\mathbf{g}^*$.*

(b) (i) *The set of regular elements $\sigma \in G$ is open and dense in G.*
  (ii) *Let σ be regular in G, $r = r_\sigma(G)$ and let $\hat{\sigma}$ be a regular element of σG^{id}. Then $r_{\hat{\sigma}}(G) = r$.*
  (iii) *Let $\sigma \in G^{\mathrm{id}}$ be regular in G, and let $\xi \in \mathbf{g}$ be regular. Then $r_\xi(G) = r_\sigma(G)$.*

(c) *The set of regular pairs $(\sigma, \mu) \in (G \times \mathbf{g}^*)^c$ is open and dense in $(G \times \mathbf{g}^*)^c$.*

Proof.

(a) Part (a) was proved by Duflo and Vergne, see [12].
(b) Part (b)(i) can be shown in the same way. Let $\sigma \in G$ be regular, $r = r_\sigma(G)$, and let V_σ be the irreducible component of $V = G$ containing σ. The condition $\dim(\mathbf{g}_{\hat{\sigma}}) = \dim\ker(\mathrm{Ad}_{\hat{\sigma}} - \mathrm{id}) > r$ is an algebraic condition on $\hat{\sigma} \in G$ since it is equivalent to

the conditions that the determinants of all (r, r)-minors of the matrix $\mathrm{Ad}_{\hat{\sigma}} - \mathrm{id}$ vanish. Moreover, these conditions define a proper subset of V_σ which is a subvariety of V_σ. So the complement of this subvariety is Zariski-open and, due to the irreducibility of V_σ, open and dense in V_σ. This proves (i). Part (ii) follows from the fact that by Lemma 3.4 the subvariety V_σ contains the connected component σG^{id} of σ in G. From $r_\xi(G) = r_{\exp(\xi)}(G)$ for $\xi \in \mathbf{g}$ small we obtain (iii).

(c) Part (c) follows from the corresponding statement [13, Theorem I.2.4] for general varieties mentioned above. □

Note that $r_\sigma(G)$ does depend on the connected component containing σ. Let for example, $G = \mathrm{O}(2)$. Then for $\sigma \in \mathrm{O}(2) \setminus \mathrm{SO}(2)$ we have $r_\sigma(G) = 0$ and for $\sigma \in \mathrm{SO}(2)$ we have $r_\sigma(G) = 1$. By definition, minimal drift–momentum pairs are also open and dense in $(G \times \mathbf{g}^*)^{\mathrm{c}}$. We conclude this section with the following remarks.

Remark 3.6.

(a) We call a point p of an algebraic variety $V = V(f)$ for a vector of polynomials f *minimal* (with respect to f) if $\ker \mathrm{D}f(p)$ has locally minimal dimension in V. If the components f_i, $i = 1, \dots, n$, of the vector $f = (f_1, \dots, f_n)$ of polynomials generate the ideal $I(V) \subseteq \mathbb{R}[x]$ of all polynomials which vanish on V then for every $p \in V$ the dimension of $\ker \mathrm{D}\hat{f}(p)$, with $\hat{f} = (\hat{f}_1, \dots, \hat{f}_{\hat{n}})$, $\hat{f}_i \in I(V)$, $i = 1, \dots, \hat{n}$, has a minimum for $\hat{f} = f$, and in this case $\dim \ker \mathrm{D}f(p)$ is called the *embedding dimension* $\mathrm{edim}_p\, V$ of V at p [6]. For every smooth point p we have $\mathrm{edim}_p\, V \geq \dim_p V$, but in general smooth points of varieties are not minimal, and equality does not in general hold, even if V is irreducible and the f_i, $i = 1, \dots, n$, generate $I(V)$, cf. [4, Example 3.3.11(b)].

Part (b) below shows that a drift–momentum pair $(\sigma, \mu) \in (G \times \mathbf{g}^*)^{\mathrm{c}}$ is minimal in the sense of Definition 3.1 if it is a minimal point of the variety $V(f) = (G \times \mathbf{g}^*)^{\mathrm{c}}$ with respect to $f = \pi - \mathrm{id}$ with π from (2.18). This supplements Remark 3.2.

(b) For $(\sigma, \mu) \in (G \times \mathbf{g}^*)^{\mathrm{c}}$ we have

$$\dim G_{(\sigma,\mu)} + \dim G = \dim \ker(\mathrm{D}\pi(\sigma, \mu) - P_{\mathbf{g}^*}),$$

where π is from (2.18) and $P_{\mathbf{g}^*} : G \times \mathbf{g}^* \to \mathbf{g}^*$, $P_{\mathbf{g}^*}(\sigma, \mu) = \mu$, is the projection of (σ, μ) onto its second component.

Proof. Let $(\hat{\sigma}, \hat{\mu}) \in \ker(\mathrm{D}\pi(\sigma, \mu) - P_{\mathbf{g}^*})$, and $\hat{\sigma} = \hat{\xi}\sigma$, $\hat{\xi} \in \mathbf{g}$. Then $-\mathrm{ad}^*_{\hat{\xi}}\, \mu + (\mathrm{Ad}^*_{\sigma^{-1}} - \mathrm{id})\hat{\mu} = 0$. Hence $(\mathrm{Ad}^*_{\sigma^{-1}} - \mathrm{id})\hat{\mu} \in \mathbf{g}\mu = \mathcal{T}_\mu G\mu$. Using the decomposition $\mathbf{g} = \mathbf{g}_\mu \oplus \mathbf{n}_\mu$ and the fact that $\mathrm{ann}(\mathbf{g}_\mu) = \mathbf{g}\mu$ we can decompose $\mathbf{g}^* = \mathbf{g}\mu \oplus \mathrm{ann}(\mathbf{n}_\mu)$ and therefore we can write $\hat{\mu} = \mathrm{ad}^*_\eta\, \mu + \hat{\nu}$ with $\hat{\nu} \in \mathrm{ann}(\mathbf{n}_\mu)$ and $\eta \in \mathbf{n}_\mu$. The condition $(\mathrm{Ad}^*_{\sigma^{-1}} - \mathrm{id})\hat{\mu} \in \mathbf{g}\mu$ is equivalent to $(\mathrm{Ad}^*_{\sigma^{-1}} - \mathrm{id})\hat{\nu} \in \mathbf{g}\mu$. Since $\sigma \in G_\mu$ we conclude that Ad_σ maps $\mathbf{g}_\mu$ into itself and so, the condition $(\mathrm{Ad}^*_{\sigma^{-1}} - \mathrm{id})\hat{\nu} \in \mathbf{g}\mu$ is equivalent to $(\mathrm{Ad}^{\mu,*}_{\sigma^{-1}} - \mathrm{id})\hat{\nu} = 0$. The proof now follows from the facts that $\dim \ker(\mathrm{Ad}^{\mu,*}_{\sigma^{-1}} - \mathrm{id}) = \dim \mathbf{g}_{(\sigma,\mu)}$, that $\hat{\xi}$ is only determined modulo $\mathbf{g}_\mu$ and that $\mathrm{ad}^*_\eta\, \mu \in \mathrm{ann}(\mathbf{g}_\mu)$ can be chosen arbitrarily. □

(c) In [4, Definition 3.3.1] points $p \in V$ with $\mathrm{edim}_p V = \dim_p V$ are called regular (non-singular). From [4, Propositions 3.3.9 and 3.3.10, Example 3.3.11(b)] it follows that this is a stronger condition than our notion of regularity.
(d) The *stratification theorem for algebraic varieties* [4, Theorem 2.3.6; 13, Theorem I.2.7] states that every algebraic variety can be decomposed into a finite number of submanifolds, called *strata*, which are semi-algebraic sets, and that this decomposition is locally finite and can be chosen to be Whitney-regular. This theorem is needed to develop a persistence theory for relative periodic orbits with arbitrary drift–momentum pairs—an open problem which is beyond the scope of this paper.

3.3. Local structure of the space of drift–momentum pairs

This section contains some technical lemmata which are needed later in the proofs. It can be skipped at first reading.

The following two lemmata are needed to study the local structure of the space of drift–momentum pairs. The first lemma shows that $\mathbf{g}_\mu^*$ can be interpreted as a section transverse to a coadjoint orbit $G\mu \subset \mathbf{g}^*$ at $\mu \in \mathbf{g}^*$.

Lemma 3.7 ([40, Lemma 2.1(a)]). *Let $\mu \in \mathbf{g}^*$ and $\mathbf{g}_\mu \oplus \mathbf{n}_\mu = \mathbf{g}$. The space $\mu + \mathbf{g}_\mu^*$ with $\mathbf{g}_\mu^* \simeq \mathrm{ann}(\mathbf{n}_\mu)$ is a section transverse to $G\mu$ at μ. More precisely, there are neighbourhoods U of μ in $\mathbf{g}^*$, $U_G(\mathrm{id})$ of id in G and $U_{\mathbf{g}_\mu^*}(0)$ of 0 in $\mathbf{g}_\mu^*$ such that*

$$\text{for all } \hat{\mu} \in U \text{ there are } g \in U_G(\mathrm{id}), \qquad \nu \in U_{\mathbf{g}_\mu^*}(0) \quad \text{such that } \hat{\mu} = g(\mu + \nu), \tag{3.1}$$

and (g, ν) is locally unique modulo the action of $U_G(\mathrm{id}) \cap Z_{\mu,\nu}$ given by

$$g_\mu(g, \nu) := (g g_\mu^{-1}, g_\mu \circ \nu), \qquad g_\mu \circ \nu := g_\mu(\mu + \nu) - \mu,$$
$$g_\mu \in U_G(\mathrm{id}) \cap Z_{\mu,\nu}. \tag{3.2}$$

If μ is split then $g_\mu \circ \nu = g_\mu \nu$. If we choose $g = \exp(\chi)$, where $\chi \in \mathbf{n}_\mu$, $\chi \approx 0$, then χ and ν are locally unique.

In the following lemma we describe the set $\tilde{Z}_{\mu,\nu}$ defined in (2.16) in a neighbourhood of $\sigma \in G_\mu$ for given $\mu \in \mathbf{g}^*$ and $\nu \in \mathbf{g}_\mu^*$. For any set $A \subseteq G$ and $\sigma \in A$ denote by A^σ the connected component of A containing σ.

Lemma 3.8. *Let $(\sigma, \mu) \in (G \times \mathbf{g}^*)^{\mathrm{c}}$ and $\mathbf{g}_\mu \oplus \mathbf{n}_\mu = \mathbf{g}$.*

(a) *Let $U_G(\mathrm{id})$ be a sufficiently small neighbourhood of id in G and $U_G(\sigma) = \sigma U_G(\mathrm{id})$. Then the set $\tilde{Z}_{\mu,\nu} \cap U_G(\sigma)$, where $\nu \in \mathbf{g}_\mu^* \simeq \mathrm{ann}(\mathbf{n}_\mu)$ is small, is a manifold of dimension $\dim G_\mu$. Moreover, there is a smooth function $e_{(\sigma,\mu)} : U_{\mathbf{g}_\mu \oplus \mathbf{g}_\mu^*}(0) \to U_G(\mathrm{id})$ with*

$$\sigma e_{(\sigma,\mu)}(\cdot, \nu) \in \tilde{Z}_{\mu,\nu}, \qquad e_{(\sigma,\mu)}(\xi, 0) = \exp(\xi) \quad \text{for all; } \xi \in \mathbf{g}_\mu,$$

which is defined by

$$e_{(\sigma,\mu)}(\chi, \nu) = \exp(\chi + \eta^e_{(\sigma,\mu)}(\chi, \nu)).$$

Here $\eta^e_{(\sigma,\mu)} : \mathbf{g}_\mu \oplus \mathbf{g}^*_\mu \to \mathbf{n}_\mu$ *is smooth in* (σ, χ, ν) *and satisfies*

$$P_{\mathrm{ann}(\mathbf{g}_\mu)}(\sigma \exp(\chi + \eta^e_{(\sigma,\mu)}(\chi, \nu))(\mu + \nu) - \mu) = 0, \tag{3.3}$$

and

$$\eta^e_{(\sigma,\mu)}(\chi, 0) = 0, \qquad \mathrm{D}_\chi \eta^e_{(\mathrm{id},\mu)}(\chi, \nu)|_{\chi=0} = \eta_\mu(\nu). \tag{3.4}$$

(b) *If* μ *is* σ*-split and* $\mathrm{ann}(\mathbf{n}_\mu)$ *invariant under* Ad^*_σ *and* G^{id}_μ *then* $\eta^e_{(\sigma,\mu)} \equiv 0$ *and* $e_{(\sigma,\mu)}(\cdot, \nu) = \exp(\cdot)$.

(c) *If* $K \subseteq G_\mu$ *is a subgroup of* G_μ *such that* $\mathbf{n}_\mu$ *is* K*-invariant then for* $g \in K$ *and* $g_\mu \in \tilde{Z}_{\mu,\nu}$ *we have*

$$(\tilde{Z}_{\mu,\nu})^{gg_\mu} = g(\tilde{Z}_{\mu,\nu})^{g_\mu}, \tag{3.5}$$

and for $g_\mu \in G_\mu$ *we get*

$$e_{(gg_\mu,\mu)}(\chi, \nu) = e_{(g_\mu,\mu)}(\chi, \nu). \tag{3.6}$$

In particular, for $\sigma = \alpha \exp(\xi) \in G_\mu$ *as in (2.3) this gives*

$$e_{(\sigma,\mu)}(\chi, \nu) = e_{(\exp(\xi),\mu)}(\chi, \nu). \tag{3.7}$$

(d) *If* $\mu \in \mathbf{g}^*$ *is regular then* $Z_{\mu,\nu} = G^{\mathrm{id}}_{\mu+\nu}$ *is Abelian and* $g_\mu \circ \nu = \nu$ *for each small* $\nu \in \mathrm{ann}(\mathbf{n}_\mu)$ *and* $g_\mu \in Z_{\mu,\nu}$.

Proof.

(a) Part (a) follows from the implicit function theorem applied to $F(\eta, \chi, \nu) = 0$ with $F : \mathbf{n}_\mu \oplus \mathbf{g}_\mu \oplus \mathbf{g}^*_\mu \to \mathbf{g}\mu$ defined by

$$F(\eta, \chi, \nu) = P_{\mathrm{ann}(\mathbf{g}_\mu)}(\sigma \exp(\chi + \eta)(\mu + \nu) - \mu).$$

Here we use that $\mathrm{D}_\eta F(0)$, given by $\mathrm{D}_\eta F(0)\hat{\eta} = -P_{\mathrm{ann}(\mathbf{g}_\mu)} \mathrm{Ad}^*_{\sigma^{-1}} \mathrm{ad}^*_{\hat{\eta}} \mu$, has image $\mathrm{im}\, \mathrm{D}_\eta F(0) = P_{\mathrm{ann}(\mathbf{g}_\mu)} \mathrm{Ad}^*_{\sigma^{-1}} \mathbf{g}\mu = P_{\mathrm{ann}(\mathbf{g}_\mu)} \mathbf{g}\mu = \mathbf{g}\mu$ and therefore full rank. The properties (3.4) of $\eta^e_{(\sigma,\mu)}$ follow from the definition of $\eta^e_{(\sigma,\mu)}$, see (3.3), and the definition of η_μ, cf. (2.10).

(b) Is clear.

(c) To prove (3.5) and (3.6) apply $g \in K \subseteq G_\mu$, to (3.3) and use that $P_{\mathrm{ann}(\mathbf{g}_\mu)}$ commutes with g. Let $\sigma = \alpha \exp(\xi) \in G_\mu$ as in (2.3). Since $\mathbf{n}_\mu$ is Ad_α-invariant we can deduce (3.7) from (3.6).

(d) Let μ be minimal. Then $G_{\hat{\mu}}$ has locally constant dimension for $\hat{\mu}$ in a small neighbourhood of μ in $\mathbf{g}^*$, and therefore, since $G^{\mathrm{id}}_{\mu+\nu} \subseteq Z_{\mu,\nu}$ for $\nu \in \mathrm{ann}(\mathbf{n}_\mu)$ small, and since both, $G^{\mathrm{id}}_{\mu+\nu}$ and by part (a) also $Z_{\mu,\nu}$, have dimension $\dim \mathbf{g}_\mu$, we get $Z_{\mu,\nu} = G^{\mathrm{id}}_{\mu+\nu}$, see also [33]. Hence $g_\mu \circ \nu = \nu$ for $g_\mu \in Z_{\mu,\nu}$, $\nu \in \mathrm{ann}(\mathbf{n}_\mu)$ small. The fact that then $G^{\mathrm{id}}_{\mu+\nu}$ is Abelian is due to Duflo and Vergne [12] and it is also a consequence of the equality $\mathrm{ad}^{\mu,*}_\xi \equiv 0$ for $\xi \in \mathbf{g}_\mu$ which follows from differentiating $g_\mu \circ \nu = \nu$ at $g_\mu = \mathrm{id}$. □

Remark 3.9.

(a) The variety $(G \times \mathbf{g}^*)^c$ is invariant under the G-action $(\sigma, \mu) \to (g\sigma g^{-1}, g\mu)$, $g \in G$, and invariant under the transformation $(\sigma, \mu) \to (\sigma^{-1}, \mu)$.

(b) Note that with $(\sigma, \mu) \in (G \times \mathbf{g}^*)^c$ also $(G_\mu, \mu) \subseteq (G \times \mathbf{g}^*)^c$. Moreover, $G(\sigma, \mu) \subseteq (G \times \mathbf{g}^*)^c$ and is disjoint from (G_μ, μ) if we replace G by $\{\exp(\eta), \eta \in U_{\mathbf{n}_\mu}(0)\}$, where $U_{\mathbf{n}_\mu}(0)$ is a sufficiently small neighbourhood of 0 in $\mathbf{n}_\mu$. Since $\mathbf{n}_\mu$ has dimension $\dim G - \dim G_\mu$ we see that $r_{(\sigma,\mu)}(G)$ is the non-trivial term in the dimension formula (see Definition 3.1(c))

$$\dim_{(\sigma,\mu)}(G \times \mathbf{g}^*)^c = r_{(\sigma,\mu)}(G) + \dim G.$$

We will make this interpretation of $r_{(\sigma,\mu)}(G)$ more precise in the next lemma.

The following technical lemma which analyses the local structure of the space of drift–momentum pairs is analogous to the corresponding results of Patrick [30] and Wulff [39] on velocity–momentum pairs $(\xi, \mu) \in (\mathbf{g} \oplus \mathbf{g}^*)^c$.

Lemma 3.10. *Let* $(\sigma, \mu) \in (G \times \mathbf{g}^*)^c$, $\mathbf{g} = \mathbf{g}_\mu \oplus \mathbf{n}_\mu$, *and identify* $\mathbf{g}_\mu^* = \mathrm{ann}(\mathbf{n}_\mu)$.

(a) *Using the section* $\mu + \mathbf{g}_\mu^*$ *transverse to* $G\mu$ *at* μ *from* Lemma 3.7 *we get the following parameterization of* $(G \times \mathbf{g}^*)^c$ *in a neighbourhood* $U_{(G\times\mathbf{g}^*)^c}(\sigma, \mu)$ *of* (σ, μ) *in* $(G \times \mathbf{g}^*)^c$:

$$U_{(G\times\mathbf{g}^*)^c}(\sigma, \mu) = \{g(\sigma e_{(\sigma,\mu)}(\chi, \nu), \mu + \nu), g \in U_G(\mathrm{id}),\\ (\chi, \nu) \in U_{\mathbf{g}_\mu \oplus \mathbf{g}_\mu^*}(0) \;\text{ satisfies }\; \pi_{(\sigma,\mu)}(\chi, \nu) = \nu\}.$$

Here $U_G(\mathrm{id})$, $U_{\mathbf{g}_\mu\oplus\mathbf{g}_\mu^*}(0)$ *are neighbourhoods of* id *in* G, 0 *in* $\mathbf{g}_\mu \oplus \mathbf{g}_\mu^*$, *respectively*, $\pi_{(\sigma,\mu)}$ *is defined by*

$$\pi_{(\sigma,\mu)} : \mathbf{g}_\mu \times \mathbf{g}_\mu^* \to \mathbf{g}_\mu^*, \qquad \pi_{(\sigma,\mu)}(\chi, \nu) = \sigma e_{(\sigma,\mu)}(\chi, \nu)(\mu + \nu) - \mu, \tag{3.8}$$

and (g, χ, ν) *are locally unique modulo the action of* $U_G(\mathrm{id}) \cap Z_{\mu,\nu}$ *given by*

$$g_\mu(g, \chi, \nu) := (gg_\mu^{-1}, g_\mu \circ_{(\sigma,\mu,\nu)} \chi, g_\mu \circ \nu), \quad g_\mu \in U_G(\mathrm{id}) \cap Z_{\mu,\nu}, \tag{3.9}$$

where $g_\mu \circ_{(\sigma,\mu,\nu)} \chi \in \mathbf{g}_\mu$ *is defined by the equation*

$$g_\mu \sigma e_{(\sigma,\mu)}(\chi, \nu) g_\mu^{-1} = \sigma e_{(\sigma,\mu)}(g_\mu \circ_{(\sigma,\mu,\nu)} \chi, g_\mu \circ \nu),$$

$Z_{\mu,\nu}$ *is as in* (2.16), *and* $g_\mu \circ \nu$ *is given by* (3.2). *Moreover* $(\pi - P_{\mathbf{g}^*})^{-1}(0) = (G \times \mathbf{g}^*)^c$ *is locally a manifold near* $(\sigma, \mu) \in (G \times \mathbf{g}^*)^c$ *if and only if* $(\pi_{(\sigma,\mu)} - P_{\mathbf{g}_\mu^*})^{-1}(0)$ *is locally a manifold near* 0. *If* μ *is* σ*-split we have*

$$g_\mu \circ \nu = g_\mu \nu, \qquad g_\mu \circ_{(\sigma,\mu,\nu)} \chi = g_\mu \chi, \qquad \pi_{(\sigma,\mu)}(\chi, \nu) = \sigma \exp(\chi)\nu. \tag{3.10}$$

(b) *Let* $(\sigma, \mu) \in (G \times \mathbf{g}^*)^c$ *be regular and let* $r = r_{(\sigma,\mu)}(G)$ *be such that* $\dim_{(\sigma,\mu)}(G \times \mathbf{g}^*)^c = r + \dim G$. *Then* $r \le r_\sigma(G_\mu)$ *and near* $(\chi, \nu) = 0$ *the set* $(\pi_{(\sigma,\mu)} - P_{\mathbf{g}_\mu^*})^{-1}(0)$ *can be parameterized by* $(\chi, \nu(\chi, \lambda))$, *where* (χ, λ) *lies in a neighbourhood* $U_{\mathbf{g}_\mu\oplus\mathbb{R}^r}(0)$ *of* 0 *in* $\mathbf{g}_\mu \oplus \mathbb{R}^r$, $\nu : U_{\mathbf{g}_\mu\oplus\mathbb{R}^r}(0) \to \mathbf{g}_\mu^*$ *is smooth*, $\nu(0) = 0$, $\mathrm{D}_\chi\nu(0) = 0$ *and the columns of*

$D_\lambda \nu(0)$ *span an* r*-dimensional subspace of* $\mathrm{Fix}_{\mathbf{g}_\mu^*}(\mathrm{Ad}_\sigma^{\mu,*})$. *If* μ *is* σ*-split then* $\nu(\chi, \lambda)$ *is linear in* λ *and* $r = r_\sigma(G_\mu)$.

Proof. (a) follows from Lemmas 3.7 and 3.8. To prove part (b) let $(\sigma, \mu) \in (G \times \mathbf{g}^*)^{\mathrm{c}}$ be regular with $r = r_{(\sigma,\mu)}(G)$. Then by part (a) $(\pi_{(\sigma,\mu)} - P_{\mathbf{g}_\mu^*})^{-1}(0) \subseteq \mathbf{g}_\mu \oplus \mathbf{g}_\mu^*$ is a manifold near 0 with $\dim_0(\pi_{(\sigma,\mu)} - P_{\mathbf{g}_\mu^*})^{-1}(0) = r + \dim \mathbf{g}_\mu$. Since $(\pi_{(\sigma,\mu)} - P_{\mathbf{g}_\mu^*})^{-1}(0)$ clearly contains the vectorspace $\mathbf{g}_\mu \oplus \{0\} \subseteq \mathbf{g}_\mu \oplus \mathbf{g}_\mu^*$ we can locally parameterize $(\pi_{(\sigma,\mu)} - P_{\mathbf{g}_\mu^*})^{-1}(0)$ as $(\chi, \nu(\chi, \lambda))$, where $\chi \in \mathbf{g}_\mu$ and $\lambda \in \mathbb{R}^r$ are small, $\nu(0) = 0$, $D_\chi \nu(0) = 0$, and the rank of $D_\lambda \nu(0)$ is r. Differentiating $(\pi_{(\sigma,\mu)} - P_{\mathbf{g}_\mu^*})(\chi, \nu(\chi, \lambda)) = 0$ at 0 we get $(\mathrm{Ad}_{\sigma^{-1}}^{\mu,*} - \mathrm{id})\, D_\lambda \nu(0) = 0$. As a consequence $r \le \dim \ker(\mathrm{Ad}_{\sigma^{-1}}^{\mu,*} - \mathrm{id}) = \dim \mathbf{g}_{(\sigma,\mu)}$. The last statement follows from the fact that if μ is σ-split then (3.10) holds. □

3.4. Sufficient conditions for regularity

In the following proposition we present some sufficient conditions for the regularity of drift–momentum pairs which are easy to check. In particular, we show that regularity of the momentum or the drift symmetry imply the regularity of the drift–momentum pairs. The results are similar to the case of velocity–momentum pairs of relative equilibria [30,39].

Proposition 3.11.

(a) *If* $\mu \in \mathbf{g}^*$ *is regular then* (σ, μ) *is regular and minimal for every* $\sigma \in G_\mu$, *and* $r_{(\sigma,\mu)}(G) = \dim \mathbf{g}_{(\sigma,\mu)} = \dim \mathbf{g}_{(\alpha,\mu)}$, *where* $\sigma = \alpha \exp(\xi)$ *as in* (2.3).
(b) *If* $\sigma \in G$ *is regular in* G *then* $(\sigma, \mu) \in (G \times \mathbf{g}^*)^{\mathrm{c}}$ *is regular and minimal for every* $\mu \in \mathbf{g}^*$ *with* $\mathrm{Ad}_\sigma^* \mu = \mu$, *and* $r_\sigma(G) = \dim \mathbf{g}_{(\sigma,\mu)} = r_{(\sigma,\mu)}(G)$.
(c) *If* μ *is* σ*-split and* $\mathbf{n}_\mu$ *is chosen to be invariant under* G_μ^{id} *and* Ad_σ (*where* $\sigma \in G_\mu$) *then the following conditions are equivalent*:
 (i) σ *is regular in* G_μ;
 (ii) $(\sigma, \mu) \in (G \times \mathbf{g}^*)^{\mathrm{c}}$ *is regular*;
 (iii) $(\sigma, \mu) \in (G \times \mathbf{g}^*)^{\mathrm{c}}$ *is minimal.*

 Moreover, if μ *is* σ*-split and* (σ, μ) *regular then* $r_{(\sigma,\mu)}(G) = \dim \mathbf{g}_{(\sigma,\mu)}$.
(d) *With* $(\sigma, \mu) \in (G \times \mathbf{g}^*)^{\mathrm{c}}$ *also* (σ^{-1}, μ) *is regular and* $r_{(\sigma,\mu)}(G) = r_{(\sigma^{-1},\mu)}(G)$.

Proof.

(a) If μ is minimal then from (3.5), (3.7) and (3.8) and Lemma 3.8(d) we get $\pi_{(\sigma,\mu)}(\chi, \nu) = \alpha\nu$. So close to (σ, μ) the variety $(G \times \mathbf{g}^*)^{\mathrm{c}}$ has dimension

$$\dim_{(\sigma,\mu)}(G \times \mathbf{g}^*)^{\mathrm{c}} = \dim \mathbf{g}_{(\alpha,\mu)} + \dim G.$$

Since G_μ^{id} is Abelian for μ minimal by Lemma 3.8(d) we have $\mathrm{Ad}_{\exp(\xi)}\, \chi = \chi$ for each $\chi \in \mathbf{g}_\mu$ and therefore $\mathbf{g}_{(\sigma,\mu)} = \mathbf{g}_{(\alpha,\mu)}$. This shows that for every $\sigma \in G_\mu$ the drift–momentum pair $(\sigma, \mu) \in (G \times \mathbf{g}^*)^{\mathrm{c}}$ is regular with $r_{(\sigma,\mu)}(G) = \dim \mathbf{g}_{(\sigma,\mu)}$. Since drift–momentum pairs $(\hat\sigma, \hat\mu) \in (G \times \mathbf{g}^*)^{\mathrm{c}}$ close to (σ, μ) are also regular with

$r_{(\hat\sigma,\hat\mu)}(G) = r_{(\sigma,\mu)}(G) = \dim \mathbf{g}_{(\sigma,\mu)}$ and by Lemma 3.10(b) we have $\dim \mathbf{g}_{(\sigma,\mu)} = r_{(\hat\sigma,\hat\mu)}(G) \le \dim \mathbf{g}_{(\hat\sigma,\hat\mu)}$ we conclude that (σ,μ) is also minimal.

(b) If σ is minimal then $G_{\hat\sigma}$ has locally constant dimension for $\hat\sigma$ in a small neighbourhood of σ in G. So close to $\{\sigma\} \times \mathrm{Fix}_{\mathbf{g}^*}(\mathrm{Ad}^*_\sigma) \subseteq (G \times \mathbf{g}^*)^{\mathrm{c}}$ the variety $(G \times \mathbf{g}^*)^{\mathrm{c}}$ is a trivial bundle with fibre $\mathrm{Fix}_{\mathbf{g}^*}(\mathrm{Ad}^*_{\hat\sigma})$ isomorphic to $\mathrm{Fix}_{\mathbf{g}^*}(\mathrm{Ad}^*_\sigma)$ over a neighbourhood $\hat\sigma \approx \sigma$ of σ in G, and the dimension of $(G \times \mathbf{g}^*)^{\mathrm{c}}$ near $\{\sigma\} \times \mathrm{Fix}_{\mathbf{g}^*}(\mathrm{Ad}^*_\sigma)$ is therefore $\dim G + \dim G_\sigma$. This shows that for every $\mu \in \mathbf{g}^*$ with $\mathrm{Ad}^*_\sigma \mu = \mu$ the drift–momentum pair $(\sigma,\mu) \in (G \times \mathbf{g}^*)^{\mathrm{c}}$ is regular with $r_{(\sigma,\mu)}(G) = r_\sigma(G)$. Since $r_{(\sigma,\mu)}(G) \le \dim \mathbf{g}_{(\sigma,\mu)}$ by Lemma 3.10(b) and $\dim \mathbf{g}_{(\sigma,\mu)} \le r_\sigma(G) = \dim \mathbf{g}_\sigma$ we have $r_{(\sigma,\mu)}(G) = \dim \mathbf{g}_{(\sigma,\mu)}$.

(c) If μ is σ-split then by (3.10) we have $\pi_{(\sigma,\mu)}(\chi,\nu) = \sigma\exp(\chi)\nu$ where $\nu \in \mathbf{g}^*_\mu$, $\chi \in \mathbf{g}_\mu$. We first show that (i) implies (ii). Let σ be minimal in G_μ. From part (b) we see that $(\sigma, 0)$ is a regular point of $(G_\mu \times \mathbf{g}^*_\mu)^{\mathrm{c}}$ with $r_{(\sigma,0)}(G_\mu) = \dim \mathbf{g}_{(\sigma,\mu)}$. From Lemma 3.10(a) we conclude that (σ,μ) is a regular point of $(G \times \mathbf{g}^*)^{\mathrm{c}}$ with $r_{(\sigma,\mu)}(G) = \dim \mathbf{g}_{(\sigma,\mu)}$.

Now we show that (ii) implies (iii). If μ is σ-split and (σ,μ) a regular point of $(G \times \mathbf{g}^*)^{\mathrm{c}}$ with $\dim_{(\sigma,\mu)}(G \times \mathbf{g}^*)^{\mathrm{c}} = \dim G + r$ then by Lemma 3.10(b) the variety $(G_\mu \times \mathbf{g}^*_\mu)^{\mathrm{c}}$ is a manifold near $(\sigma,0) \in (G_\mu \times \mathbf{g}^*_\mu)^{\mathrm{c}}$ of dimension $r + \dim G_\mu$ with $r = r_\sigma(G_\mu) = \dim \mathbf{g}_{(\sigma,\mu)}$. Now let $(\hat\sigma,\hat\mu) \in (G \times \mathbf{g}^*)^{\mathrm{c}}$ be close to (σ,μ). Since $(G \times \mathbf{g}^*)^{\mathrm{c}}$ is a manifold near (σ,μ) we know that $r_{(\hat\sigma,\hat\mu)}(G) = r$, with $r = \dim \mathbf{g}_{(\sigma,\mu)}$. By Lemma 3.10(b) we moreover have $r_{(\hat\sigma,\hat\mu)}(G) \le \dim \mathbf{g}_{(\hat\sigma,\hat\mu)}$. Hence $\dim \mathbf{g}_{(\sigma,\mu)} \le \dim \mathbf{g}_{(\hat\sigma,\hat\mu)}$ so that (σ,μ) is minimal.

Finally we show that (iii) implies (i). If (σ,μ) is a minimal drift–momentum pair then $\dim \mathbf{g}_{(\hat\sigma,\mu)} = \dim \mathbf{g}_{(\sigma,\mu)}$ for $\hat\sigma \in G_\mu$ close to σ, so that σ is minimal in G_μ.

(d) Let $(\sigma,\mu) \in (G \times \mathbf{g}^*)^{\mathrm{c}}$ be regular. Then by Lemma 3.10(b) the variety $(G \times \mathbf{g}^*)^{\mathrm{c}}$ can be smoothly parameterized as $g(\sigma(\chi,\lambda), \mu(\chi,\lambda))$, where $\lambda \in \mathbb{R}^r$, $r = r_{(\sigma,\mu)}(G)$, $g \in G$, $\sigma(\chi,\lambda) = \sigma e_{(\sigma,\mu)}(\chi, \nu(\chi,\lambda))$, and $\mu(\chi,\lambda) = \mu + \nu(\chi,\lambda)$. Since by Remark 3.9(a) with any (σ,μ) also $(\sigma^{-1},\mu) \in (G \times \mathbf{g}^*)^{\mathrm{c}}$ we see that near (σ^{-1},μ) the variety $(G \times \mathbf{g}^*)^{\mathrm{c}}$ is parameterized smoothly by $g((\sigma(\chi,\lambda))^{-1}, \mu(\chi,\lambda))$. So (σ^{-1},μ) is regular and $r_{(\sigma^{-1},\mu)}(G) = r_{(\sigma,\mu)}(G)$. □

4. Persistence of relative periodic orbits

In Section 4.1 we present our main result, Theorem 4.2, on persistence of non-degenerate relative periodic orbits with drift–momentum pairs which are regular modulo isotropy to relative periodic orbits with the same reduced spatio-temporal symmetry group. This extends results of Patrick et al. [30,32,39] on relative equilibria to relative periodic orbits. Afterwards, in Section 4.2, we consider persistence to relative periodic orbits with smaller reduced spatio-temporal symmetry group.

4.1. Symmetry preserving persistence

In this section we present our main result, a persistence result for relative periodic orbits which are non-degenerate and have regular drift–momentum pairs modulo isotropy. First we define what we mean by a regular drift–momentum pair modulo isotropy.

Definition 4.1. Let $p \in \mathcal{P}$ lie on a relative periodic orbit of (2.1) with momentum $\mu = \mathbf{J}(p)$, drift symmetry $\sigma \in G_\mu$, and isotropy subgroup G_p, let $L = N(G_p)/G_p$ be the symmetry group acting on the fixed point space $\mathrm{Fix}_{\mathcal{M}}(G_p)$ and let $\mathbf{l}$ denote the Lie algebra of L. Let $\mu_{\mathrm{L}} = \mu|_{\mathbf{l}}$, where we consider $\mathbf{l}$ as Σ_n-invariant subspace of $\mathbf{g}$, see Remark 2.1, and let $\sigma_{\mathrm{L}} \in L$ represent $\sigma \in G$. We say that $\mathcal{P}$ has *regular drift–momentum pair* (*regular drift*, *regular momentum*) *modulo isotropy* if $(\sigma_{\mathrm{L}}, \mu_{\mathrm{L}}) \in (L \times \mathbf{l}^*)^{\mathrm{c}}$ is regular ($\sigma_{\mathrm{L}} \in L$ is regular, $\mu \in \mathbf{l}^*$ is regular) in which case we define $r_{(\sigma,\mu)}(G_p, G) := r_{(\sigma_{\mathrm{L}},\mu_{\mathrm{L}})}(L)$ ($r_\sigma(G_p, G) := r_{\sigma_{\mathrm{L}}}(L)$, $r_\mu(G_p, G) := r_{\mu_{\mathrm{L}}}(L)$).

Propositions 3.3 and 3.11 can be used, with G replaced by L, to compute $r_{(\sigma_{\mathrm{L}},\mu_{\mathrm{L}})}(L)$, $r_{\sigma_{\mathrm{L}}}(L)$ and $r_{\mu_{\mathrm{L}}}(L)$. Now we can formulate a persistence result for relative periodic orbits with regular drift–momentum pairs modulo isotropy.

Theorem 4.2. *Let $p = \sigma^{-1}\Phi_1(p)$ lie on a relative periodic orbit $\mathcal{P}$ of* (2.1) *which is non-degenerate and has regular drift–momentum pair (σ, μ) modulo isotropy and energy $H(p) = 0$. Then the following statements hold true*:

(a) *Let $r = r_{(\sigma,\mu)}(G_p, G)$. Then there is an $(r + 1)$-dimensional smooth family of relative periodic orbits $\mathcal{P}(\lambda, E)$, parameterized by $\lambda \in \mathbb{R}^r$ and energy E, with isotropy subgroup G_p, relative period close to* 1, *drift symmetry $\sigma(\lambda, E)$ close to σ and momentum $\mu(\lambda, E) = \mathbf{J}(p(\lambda, E))$, $p(\lambda, E) \in \mathcal{P}(\lambda, E)$, close to μ such that $p(0) = p$, $\mathcal{P}(0) = \mathcal{P}$, $\sigma(0) = \sigma$, $\mu(0) = \mu$, and there are no other relative periodic orbits with the same isotropy and relative period ≈ 1 near $\mathcal{P}$.*

(b) *Let $\mathcal{M}_{\mathrm{RPO}}$ be the submanifold of $\mathcal{M}$ formed by the relative periodic orbits from* (a) *and let $\mathcal{M}_{\mathrm{RPO}}(G_p) = \mathrm{Fix}_{\mathcal{M}_{\mathrm{RPO}}}(G_p)$. Then $\mathcal{M}_{\mathrm{RPO}}(G_p)$ is a symplectic submanifold of $\mathrm{Fix}_{\mathcal{M}}(G_p)$ if and only if μ_{L} is minimal in $\mathbf{l}^*$ and $\mathbf{l}_{(\sigma_{\mathrm{L}},\mu_{\mathrm{L}})} = \mathbf{l}_{\mu_{\mathrm{L}}}$.*

(c) *The submanifold $\mathcal{M}_{\mathrm{RPO}}$ of $\mathcal{M}$ is symplectic if and only if the assumptions of* (b) *hold and the momentum isotropy algebra $\mathbf{g}_\mu$ of the relative periodic orbit lies in the Lie algebra $\mathrm{L}N(G_p)$ of the normalizer $N(G_p)$ of G_p: $\mathbf{g}_\mu \subseteq \mathrm{L}N(G_p)$. In this case there is a relative periodic orbit for any energy–momentum pair $(\hat{E}, \hat{\mu}) \in \mathbb{R} \oplus \mathbf{g}^*$ close to $(0, \mu)$.*

Proof.

(a) For simplicity, replace $\mathcal{M}$ by $\mathrm{Fix}_{\mathcal{M}}(G_p)$ so that G and L coincide. By Proposition 2.9 the persistence problem reduces to solving the fixed point equation $\Pi(\nu) = \nu$ on $\mathbf{g}_\mu^*$, where $\Pi(\nu)$ is from (2.15). By Lemma 2.8 we can write $\Pi(\nu)$ as $\Pi(\nu) = g(\nu)(\nu+\mu)-\mu$, where $g(\nu) \in \tilde{Z}_{\mu,\nu}$ and $g(\nu) \approx \sigma^{-1}$ for ν small. Note that with $(\sigma, \mu) \in (G \times \mathbf{g}^*)^{\mathrm{c}}$ also $(\sigma^{-1}, \mu) \in (G \times \mathbf{g}^*)^{\mathrm{c}}$ is regular by Proposition 3.11(d). By Lemma 3.8(a) there is some $\xi(\nu) \approx 0$ with $g(\nu) = \sigma^{-1}e_{(\sigma^{-1},\mu)}(\xi(\nu), \nu)$ and $\xi(\nu)$ is smooth in ν. So we have $\Pi(\nu) = \pi_{(\sigma^{-1},\mu)}(\xi(\nu), \nu)$, where $\pi_{(\sigma^{-1},\mu)}$ is from (3.8). Lemma 3.10(b) implies that for each small χ there is an r-dimensional manifold $\nu(\chi, \lambda)$ of solutions to $\pi_{(\sigma^{-1},\mu)}(\chi, \nu) = \nu$. Since $\mathrm{D}_\chi \nu(0) = 0$ by Lemma 3.10(b) the equation

$$\chi = \xi(\nu(\chi, \lambda))$$

can be solved for $\chi(\lambda)$. This gives an r-dimensional family $\nu(\lambda) := \nu(\chi(\lambda), \lambda)$ of (2.15) and with Proposition 2.9 proves (a).

(b) To simplify notation replace $\mathcal{M}$ by $\mathrm{Fix}_{\mathcal{M}}(G_p)$ and G by L so that G_p is trivial. By the tangent space decomposition of Theorem 2.2(a) we have $\mathcal{T}_p\mathcal{M} = \mathcal{T} \oplus \mathcal{N}$, with $\mathcal{T} = \mathcal{T}_0 \oplus \mathcal{T}_1 \oplus \mathcal{T}_2$, $\mathcal{N} = \mathcal{N}_0 \oplus \mathcal{N}_1 \oplus \mathcal{N}_2$, where $\mathcal{T}_0 \oplus \mathcal{T}_1 \simeq \mathbf{g}$, $\mathcal{T}_1 \simeq \mathbf{g}/\mathbf{g}_\mu$, $\mathcal{T}_0 \simeq \mathbf{g}_\mu$, $\mathcal{N}_0 \simeq \mathbf{g}_\mu^*$ and $\mathcal{T}_2 \oplus \mathcal{N}_2$, $\mathcal{N}_1$ and $\mathcal{T}_0 \oplus \mathcal{N}_0$ are symplectic. So we get

$$\mathcal{T}_p\mathcal{M}_{\mathrm{RPO}} \simeq \mathcal{T}_0 \oplus \mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \begin{pmatrix} \mathrm{id} & \\ & \mathrm{D}w(0) \end{pmatrix} \mathrm{Im}\,\mathrm{D}\nu(0) \oplus \mathcal{N}_2, \tag{4.1}$$

where we used bundle coordinates on the right-hand side and $\mathrm{Im}\,\mathrm{D}\nu(0)$ denotes the image of $\mathrm{D}\nu(0)$ in $\mathcal{N}_0 \simeq \mathbf{g}_\mu^*$. Since $\mathrm{Im}\,\mathrm{D}\nu(0) \subseteq \mathrm{Fix}_{\mathbf{g}_\mu^*}(\mathrm{Ad}_\sigma^{\mu,*})$ by Lemma 3.10(b) and since $\mathcal{T}_1$, $\mathcal{N}_1$, $\mathcal{T}_0 \oplus \mathcal{N}_0$ and $\mathcal{T}_2 \oplus \mathcal{N}_2$ are symplectic, the last two spaces with standard symplectic form (by Theorem 2.2(a)) the space $\mathcal{T}_p\mathcal{M}_{\mathrm{RPO}}$ is symplectic if and only if $\mathbf{g}_\mu^* = \mathrm{Fix}_{\mathbf{g}_\mu^*}(\mathrm{Ad}_\sigma^{\mu,*}) = \mathrm{Im}\,\mathrm{D}\nu(0)$. These equalities hold if and only if $\mathbf{g}_{(\sigma,\mu)} = \mathbf{g}_\mu$ and $r = \dim \mathbf{g}_\mu$. Let $(\hat\sigma, \hat\mu) \in (G \times \mathbf{g}^*)^{\mathrm{c}}$ be close to (σ, μ). By Lemma 3.10(b) we have $r_{(\hat\sigma,\hat\mu)}(G) \le r_{\hat\sigma}(G_{\hat\mu})$. Since $r = r_{(\hat\sigma,\hat\mu)}(G)$ and $r_{\hat\sigma}(G_{\hat\mu}) \le \dim \mathbf{g}_{\hat\mu}$ we get $\dim \mathbf{g}_\mu \le \dim \mathbf{g}_{\hat\mu}$. This implies that μ is minimal in $\mathbf{g}^*$.

(c) If the isotropy group G_p is continuous then (4.1) remains valid, and by Remark 2.3 the space $\mathcal{T}_0 \oplus \mathcal{N}_0$ is still a symplectic space, but now we have $\mathcal{T}_0 \simeq (\mathbf{g}_\mu/\mathbf{g}_p)$ and $\mathcal{N}_0 \simeq (\mathbf{g}_\mu/\mathbf{g}_p)^*$. As in part (b) we know that $\mathrm{Im}\,\mathrm{D}\nu(0) \subseteq \mathbf{l}_\mu^* \simeq \mathrm{Fix}_{\mathcal{N}_0}(G_p) \subseteq \mathcal{N}_0$. For $\mathcal{M}_{\mathrm{RPO}}$ to be symplectic the former two inclusions need to be equalities. We have $\mathrm{Im}\,\mathrm{D}\nu(0) = \mathbf{l}_\mu^*$ if and only if the assumptions in part (b) are satisfied. Moreover, the condition $(\mathbf{g}_\mu/\mathbf{g}_p)^* = \mathrm{Fix}_{(\mathbf{g}_\mu/\mathbf{g}_p)^*}(G_p)$ is satisfied if and only if $\mathbf{g}_\mu \subseteq \mathrm{L}N(G_p)$. □

4.2. Symmetry breaking persistence

In this section we treat persistence to relative periodic orbits with smaller reduced spatio-temporal symmetry group. We allow the bifurcating relative periodic orbit to have a *smaller isotropy group*, which is a *regular subgroup* of the isotropy group of the original relative periodic orbit (as defined in Definition 4.3), and we allow for relative period multiplying.

Let $p = \sigma^{-1}\Phi_1(p) \in \mathcal{P}$ lie on a relative periodic orbit of (2.1) with momentum $\mu = \mathbf{J}(p)$, drift symmetry $\sigma \in G_\mu$, and isotropy subgroup G_p.

Definition 4.3 (cf. [40, Definition 7.1]). The subgroup $\hat G_p$ of G_p is a *regular* subgroup of G_p if the following implication holds:

$$\chi \in \mathbf{z}(\hat G_p) \cap \mathbf{g}_p \Rightarrow \chi \in \hat{\mathbf{g}}_p.$$

Choose a regular subgroup $\hat G_p$ of G_p. To study persistence to relative periodic orbits with isotropy containing $\hat G_p$ we restrict, as in Remark 2.1, the dynamics to the flow-invariant symplectic manifold $\hat{\mathcal{M}} := \mathrm{Fix}_{\mathcal{M}}(\hat G_p)$. The symmetry group acting on the fixed point space $\hat{\mathcal{M}} = \mathrm{Fix}_{\mathcal{M}}(\hat G_p)$ is $L = N(\hat G_p)/\hat G_p$. As before denote the Lie algebra of L by $\mathbf{l}$. The fact that $\hat G_p$ is a regular subgroup of G_p means that L_p is finite. Therefore, Theorem 2.2

on the bundle structure of a neighbourhood of a relative periodic orbit can be applied to $\hat{\mathcal{M}}$. The drift symmetry σ of our relative periodic orbit $\mathcal{P}$ may not necessarily lie in L (for an example, see Section 5.3), and even if it does, it may not respect the symmetries in $\hat{G}_p$. We therefore may have to replace σ by another spatio-temporal symmetry of $\mathcal{P}$ which lies in L. This might increase the relative period. Spatio-temporal symmetries of $\mathcal{P}$ which fit to the subgroup $\hat{G}_p$ are called $\hat{G}_p$-*admissible* as detailed in the following definition.

Definition 4.4. Let $p = \sigma^{-1}\Phi_1(p) \in \mathcal{P}$, $\mu = \mathbf{J}(p)$ and let $\hat{G}_p$ be a subgroup of G_p. We say that the spatio-temporal symmetry $\hat{\sigma} = \sigma^\ell g_p \in \Sigma$, $\ell \in \mathbb{N}$, $g_p \in G_p$, of $\mathcal{P}$ (with respect to p) is $\hat{G}_p$-*admissible* if $\hat{\sigma} \in N(\hat{G}_p)$.

Note that every element $\sigma^\ell g_p$, $\ell \in \mathbb{N}$, $g_p \in G_p$, of the spatio-temporal symmetry group Σ of $\mathcal{P}$ (with respect to p) is G_p-admissible and that σ^n is $\hat{G}_p$-admissible for every subgroup $\hat{G}_p$ of G_p where $n \in \mathbb{N}$ is such that $\alpha^n = \mathrm{id}$, see (2.3). The next definition introduces the notion of non-degeneracy modulo $\hat{G}_p$.

Definition 4.5. Let $\mathcal{P}$ be a relative periodic orbit containing $p = \sigma^{-1}\Phi_1(p)$, let $\hat{G}_p$ be a regular subgroup of G_p and let $\hat{\sigma} = \sigma^\ell g_p$ be a $\hat{G}_p$-admissible drift symmetry of $\mathcal{P}$, where $\ell \in \mathbb{N}$, $g_p \in G_p$. Then $\mathcal{P}$ is called non-degenerate modulo $\hat{G}_p$, when considered as relative periodic orbit of relative period ℓ and with drift symmetry $\hat{\sigma}$, if $g_p^{-1}B_1^\ell$ does not have eigenvectors with eigenvalue 1 which lie in $\mathrm{Fix}_{\mathcal{N}_1}(\hat{G}_p)$. Here B_1 is the block in $B = \sigma^{-1}\,\mathrm{D}\Phi_1(p)$ defined in Proposition 2.5.

Now we introduce the notion of drift–momentum pairs which are regular modulo $\hat{G}_p$ by extending Definition 4.1.

Definition 4.6. Let $\mathcal{P}$, μ, σ, $\hat{G}_p$ and $L = N(\hat{G}_p)/\hat{G}_p$ be as above. Let $\hat{\sigma} = \sigma^\ell g_p$, where $\ell \in \mathbb{N}$, $g_p \in G_p$, be a $\hat{G}_p$-admissible drift symmetry of $\mathcal{P}$. Identify $\hat{\sigma}$ with $\hat{\sigma}_{\mathrm{L}} \in L$ and let $\mu_{\mathrm{L}} = \mu|_{\mathbf{l}}$ where we embed $\mathbf{l}$ into $\mathbf{g}$ as described in Remark 2.1.

We say that the drift–momentum pair $(\hat{\sigma}, \mu)$ (the drift $\hat{\sigma}$, the momentum μ) of $\mathcal{P}$ when considered as relative periodic orbit of relative period ℓ and with drift symmetry $\hat{\sigma}$ is *regular modulo* $\hat{G}_p$ if $(\hat{\sigma}_{\mathrm{L}}, \mu_{\mathrm{L}}) \in (L \times \mathbf{l}^*)^{\mathrm{c}}$ is regular ($\hat{\sigma}_{\mathrm{L}} \in L$ is regular, $\mu_{\mathrm{L}} \in \mathbf{l}^*$ is regular) in which case we define $r_{(\hat{\sigma},\mu)}(\hat{G}_p, G) := r_{(\hat{\sigma}_{\mathrm{L}},\mu_{\mathrm{L}})}(L)$ ($r_{\hat{\sigma}}(\hat{G}_p, G) := r_{\hat{\sigma}_{\mathrm{L}}}(L)$, $r_\mu(\hat{G}_p, G) := r_{\mu_{\mathrm{L}}}(L)$).

We are now ready to state our result on symmetry breaking persistence.

Theorem 4.7. *Let $p = \sigma^{-1}\Phi_1(p)$ lie on a relative periodic orbit $\mathcal{P}$ of* (2.1) *with momentum $\mu = \mathbf{J}(p)$ and energy $H(p) = 0$. Let $\hat{G}_p$ be a regular subgroup of G_p and assume that $\hat{\sigma} = \sigma^\ell g_p$ is $\hat{G}_p$-admissible where $\ell \in \mathbb{N}$, $g_p \in G_p$. Assume that $\mathcal{P}$ is non-degenerate modulo $\hat{G}_p$ when considered as relative periodic orbit of relative period ℓ with drift-symmetry $\hat{\sigma}$ and that $(\hat{\sigma}, \mu)$ is a drift–momentum pair which is regular modulo $\hat{G}_p$. Then the following statements hold true*:

(a) *Let* $r = r_{(\hat{\sigma},\mu)}(\hat{G}_p, G)$. *There is an* $(r+1)$*-dimensional smoothly parameterized family of relative periodic orbits* $\mathcal{P}(\lambda, E)$, $\lambda \in \mathbb{R}^r$, *with isotropy containing* $\hat{G}_p$, *relative period dividing* ℓ (*using the time-reparameterization of* Theorem 2.4), *and drift–momentum pair close to* $(\hat{\sigma}, \mu)$ *near* $\mathcal{P}(0) = \mathcal{P}$ *and there are no other relative periodic orbits with this property near* $\mathcal{P}$.

(b) *Let* $\mathcal{M}_{\mathrm{RPO}}$ *be the submanifold of* $\mathcal{M}$ *formed by the family of relative periodic orbits of* (a) *and let* $\mathcal{M}_{\mathrm{RPO}}(\hat{G}_p) := \mathrm{Fix}_{\mathcal{M}_{\mathrm{RPO}}}(\hat{G}_p)$. *Then* $\mathcal{M}_{\mathrm{RPO}}(\hat{G}_p)$ *is a symplectic submanifold of* $\mathrm{Fix}_{\mathcal{M}}(\hat{G}_p)$ *if and only if* μ_{L} *is minimal in* $\mathbf{l}^*$ *and* $\mathbf{l}_{(\hat{\sigma}_{\mathrm{L}},\mu_{\mathrm{L}})} = \mathbf{l}_{\mu_{\mathrm{L}}}$.

(c) *The manifold* $\mathcal{M}_{\mathrm{RPO}} \subseteq \mathcal{M}$ *is symplectic if and only if the assumptions of* (b) *hold and* $\mathbf{g}_\mu \subseteq \mathrm{L}N(\hat{G}_p)$.

Proof. The regularity assumption on $\hat{G}_p$ implies that L_p is finite so that Theorems 2.2 and 2.4 on the bundle structure near relative periodic orbits and Hamilton's equations in bundle coordinates can be applied on $\hat{\mathcal{M}} := \mathrm{Fix}_{\mathcal{M}}(\hat{G}_p)$. The equation determining relative periodic orbits near $p \in \mathcal{P}$ in $\hat{\mathcal{M}}$ therefore has the form (2.15). We treat $\mathcal{P}$ as a relative periodic orbit on $\hat{\mathcal{M}}$ with drift symmetry $\hat{\sigma}_{\mathrm{L}} \in L$ and relative period ℓ. The map Π from (2.15) has to be modified accordingly: α has to be replaced by $\alpha^\ell g_p$ and $\Psi_{1,0}$ by $\Psi_{\ell,0}$. Proposition 2.9 then still applies to this case of broken spatio-temporal symmetry, and makes the proof of the theorem analogous to the proof of Theorem 4.2. □

Remark 4.8.

(a) The assumptions of Theorem 4.7(b) are satisfied if μ is minimal in $\mathbf{l}^*$, $g_p = \mathrm{id}$ and $\ell = n$, where we decompose $\sigma = \alpha \exp(\xi)$ with $\alpha^n = \mathrm{id}$ as in (2.3). This special case was treated in [42, Corollary 4.8].

(b) [42, Theorem 4.9] is a special case of Theorem 4.7, where G_p is finite and μ is split and minimal modulo $\hat{G}_p$.

(c) Theorem 4.7 is similar to [40, Theorem 7.2] on symmetry breaking persistence of relative equilibria.

The following example is a simple illustration how Theorem 4.7 can be applied. For more examples, see Section 5.

Example 4.9. Let $G = \mathrm{O}(2)$. Then every $\mu \in \mathbf{g}^*$ is minimal. A non-degenerate relative periodic orbit $\mathcal{P}$ with momentum $\mu = 0$, zero energy and drift symmetry $\sigma \in \mathrm{O}(2) \setminus \mathrm{SO}(2)$ is a discrete rotating wave which is isolated in momentum space. If $\mathcal{P}$ is non-degenerate as 2-periodic solution then by Theorem 4.7 it persists as modulated rotating wave with relative period close to 2 for every small energy–momentum pair $(E, \mu) \neq 0$.

5. Application: oscillations of a deformable body in a fluid

In this section we illustrate how to apply the results of this paper to a specific symmetric Hamiltonian system. As our example we have chosen, as in [41], a finite dimensional model for the dynamics of a deformable body in an ideal irrotational fluid. The model extends the well known Kirchhoff model for the motion of a rigid body in a fluid [3,17,19] and the 'affine' or 'pseudo'-rigid body model used in elasticity theory [8,11,35,38].

We allow configurations that are obtained from orientation and volume preserving linear deformations and translations of a reference body. The configuration space for this system is therefore the special affine group $\mathrm{SAff}(3) = \mathrm{SL}(3) \ltimes \mathbb{R}^3$ of $\mathbb{R}^3$, where $\mathrm{SL}(3)$ is the group of invertible linear transformations of $\mathbb{R}^3$ with determinant 1 and the semi-direct product is obtained from the natural action of $\mathrm{SL}(3)$ on $\mathbb{R}^3$. The dynamics of the system are given by a Hamiltonian H on the phase space $\mathcal{T}^* \mathrm{SAff}(3)$. We assume that the reference body is a sphere. Then the deformed configurations are always ellipsoids.

5.1. Symmetries and conserved quantities

Since the reference body is spherically symmetric the Hamiltonian H is invariant under the action of $\mathrm{SO}(3)$ on $\mathcal{T}^* \mathrm{SAff}(3)$ which is induced from its natural action on the right of $\mathrm{SL}(3)$ (extended trivially to $\mathrm{SAff}(3)$):

$$B \cdot (S, s) = (SB^{-1}, s), \quad (S, s) \in \mathrm{SAff}(3), \quad B \in \mathrm{SO}(3).$$

These are the 'material' or 'body' symmetries of the system. We also assume that the system is invariant under rotations and translations of $\mathbb{R}^3$, i.e. the natural action of $\mathrm{SE}(3)$ on $\mathcal{T}^* \mathrm{SAff}(3)$ induced from its action on the left of $\mathrm{SAff}(3)$:

$$(A, a) \cdot (S, s) = (AS, a + As), \quad (S, s) \in \mathrm{SAff}(3), \quad (A, a) \in \mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3.$$

These are the 'spatial' symmetries of the system. This assumption implies that there are no external forces such as gravity acting. In particular, the body is 'neutrally buoyant' and has coincident centres of mass and buoyancy. It is natural also to assume that the system is invariant under the action of the inversion symmetry $-\mathrm{id}$ in $\mathrm{O}(3)$ acting simultaneously on the left and right of $\mathrm{SAff}(3)$. Denoting the diagonally embedded inversion operator in $\mathrm{O}(3) \times \mathrm{O}(3)$ by κ we have:

$$\kappa \cdot (S, s) = (S, -s), \quad (S, s) \in \mathrm{SAff}(3), \quad \kappa = (-\mathrm{id}, -\mathrm{id}) \in \mathrm{O}(3) \times \mathrm{O}(3).$$

Note that the action of $-\mathrm{id}$ on the left or right alone does not preserve $\mathrm{SAff}(3)$. Together the body and spatial symmetries and reflection κ generate a semi-direct product

$$G = \mathbb{Z}_2^\kappa \ltimes (\mathrm{SE}(3)_\mathrm{L} \times \mathrm{SO}(3)_\mathrm{R})$$

(here the indices L and R stand for left and right actions, respectively). This group is the symmetry group of the system.

5.2. Spherical equilibrium and non-linear normal modes

Assume, as in [41], that the spherical configuration with zero momentum $p_e = ((\mathrm{id}, 0), (0, 0))$ in $\mathrm{SAff}(3) \times \mathrm{saff}(3)^*$ is an equilibrium configuration. This has conserved momentum $\mu = 0$ and its isotropy subgroup is $G_p = \mathrm{O}(3)_\mathrm{D}$, where $\mathrm{O}(3)_\mathrm{D} = \mathbb{Z}_2^\kappa \times \mathrm{SO}(3)_\mathrm{D}$ and $\mathrm{SO}(3)_\mathrm{D} = \{((g, 0), g) \in \mathrm{SE}(3)_\mathrm{L} \times \mathrm{SO}(3)_\mathrm{R} : g \in \mathrm{SO}(3)\}$ is the diagonally embedded copy of $\mathrm{SO}(3)$ in $\mathrm{SE}(3)_\mathrm{L} \times \mathrm{SO}(3)_\mathrm{R}$.

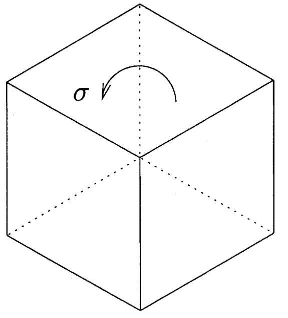

Fig. 3. Ellipsoidal oscillating normal mode.

In [41] it is shown that near a non-degenerate spherical equilibrium p_e at $\mu_e = 0$ there are three families of periodic solutions (non-linear normal modes), two of them with finite isotropy. One of the periodic solutions with finite isotropy is a ‘*cubic*’ *normal mode* with symmetry pair (Σ_n, G_p) isomorphic to

$$(\Sigma_n, G_p) = (\mathbb{T} \times \mathbb{Z}_2^\kappa, \mathbb{D}_2 \times \mathbb{Z}_2^\kappa), \quad n = 3.$$

Here $\mathbb{D}_2$ is the subgroup of $\mathrm{SO}(3)_\mathrm{D}$ consisting of rotations by π about each of three mutually perpendicular axes. The group $\mathbb{T}$ is the subgroup of order 12 in $\mathrm{SO}(3)_\mathrm{D}$ consisting of all rotations which preserve a tetrahedron. It can be generated by $\mathbb{D}_2$ together with an element α of order 3 corresponding to a rotation by $2\pi/3$ about a diagonal of the cube, see Fig. 3. The drift symmetry σ of the non-linear normal mode is $\sigma = \alpha$ and hence satisfies $n = 3$.

This periodic solution has momentum $\mu = 0$ and can be described as a ‘pulsating cube’. At all times the body is ellipsoidal (which is why G_p always contains $\mathbb{D}_2 \times \mathbb{Z}_2^\kappa$) and its principal axes have fixed directions in both body and space. However the lengths of the principal axes vary periodically, and the role of the longest principal axis is taken by each of the three in turn, with a $2\pi/3$ phase-shift between them, see Fig. 4. The spatio-temporal symmetry σ corresponds to rotating the body by $2\pi/3$ about an axis trisecting the three principal axes.

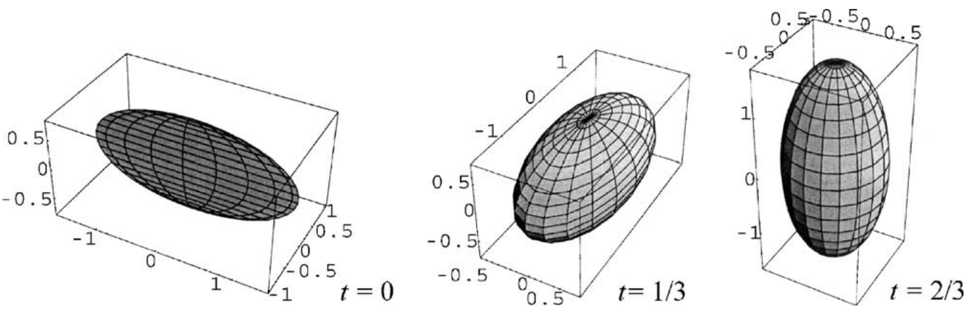

Fig. 4. Spatio-temporal symmetry σ of the pulsating cube normal mode.

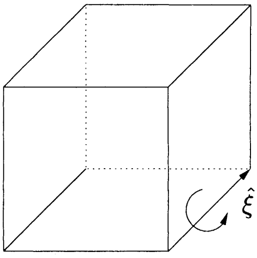

Fig. 5. Pulsating cube RPOs of Type 1.

5.3. Relative periodic orbits

Let τ be a rotation by π in $\mathbb{D}_2 = \mathrm{SO}(3)_\mathrm{D} \cap G_p$. In [41] it is shown, using the persistence result [42, Theorem 4.9] (see also Remark 4.8(b)) for minimal momenta modulo $\hat{G}_p$ for the subgroup $\hat{G}_p := \mathbb{Z}_2^\tau$ of G_p that the normal mode $\mathcal{P}$ perturbs to a four-dimensional family of relative periodic orbits with relative period 3 and isotropy $\hat{G}_p = \mathbb{Z}_2^\tau$. Here $\mathbf{l} = \mathbf{l}_\mu = \mathrm{so}(2)_\mathrm{L} \oplus \mathbb{R}_\mathrm{L} \oplus \mathrm{so}(2)_\mathrm{R}$. This is an example where $\hat{\sigma} \simeq \sigma$ is not $\hat{G}_p$-admissible, but $\hat{\sigma} \simeq \sigma^3 = \mathrm{id}$ is. The persisting relative periodic orbits rotate around and translate along one of the principal axes (the rotation axis of τ), see Fig. 5. The details from [41] can be found in Table 1. In this table $\hat{G}_p \subseteq G_p$ denotes the isotropy group of the bifurcating relative periodic orbits, $\hat{G}_p \subseteq \Sigma_n$ denotes their isotropy group, $\hat{\Sigma} \subseteq \Sigma_n$ their reduced spatio-temporal symmetry group, ℓ their relative period, $\hat{\sigma} \in G$ their drift symmetry and $\hat{\xi} = (\hat{\xi}_\mathrm{L}, \hat{\xi}_\mathrm{T}, \hat{\xi}_\mathrm{R}) \in \mathbf{g} = \mathrm{so}(3)_\mathrm{L} \oplus \mathbb{R}_3 \oplus \mathrm{so}(3)_\mathrm{R}$ the drift direction of the bifurcating relative periodic orbits. As can be seen from Table 1 this family of bifurcating relative periodic orbits contains two three-dimensional subfamilies with higher isotropy, one which consists of relative periodic orbits which rotate but do not translate (case b) and one which consists of relative periodic orbits which translate, but do not rotate (case c).

With our persistence result on symmetry breaking persistence of relative periodic orbits, Theorem 4.7, we can show that there is a second family of relative periodic orbits nearby with relative period close to one. The pair $(\sigma = \alpha, \mu = 0)$ is a regular drift–momentum pair because $\alpha \in G$ is minimal with $\mathbf{g}_\sigma = \mathrm{so}(2)_\mathrm{L} \oplus \mathbb{R} \oplus \mathrm{so}(2)_\mathrm{R}$. So by Theorem 4.2 there is a four-parameter family $\mathcal{P}(\lambda, E)$, $\lambda \in \mathbb{R}^3$, of relative periodic orbits close to $\mathcal{P}$ with relative period close to 1 and drift symmetry $\sigma(\lambda, E)$ close to α. The relative periodic orbits of this

Table 1
Symmetries of Type 1 relative periodic orbits bifurcating from the cubic oscillations [41]; τ and $\hat{\tau}$ are two different non-identity elements in $\mathbb{D}_2 = \mathrm{SO}(3)_\mathrm{D} \cap G_p$

	$\hat{G}_p$	$\hat{\Sigma}$	ℓ	$\hat{\sigma}$	$\hat{\xi}$
(a)	$\mathbb{Z}_2^\tau$	$\mathbb{Z}_2^\tau$	3	$\exp(\hat{\xi})$	$\hat{\xi}_\mathrm{R} \| \hat{\xi}_\mathrm{L} \| \hat{\xi}_\mathrm{T} \| \tau$
(b)	$\mathbb{Z}_2^\tau \times \mathbb{Z}_2^\kappa$	$\mathbb{Z}_2^\tau \times \mathbb{Z}_2^\kappa$	3	$\exp(\hat{\xi})$	$\hat{\xi}_\mathrm{R} \| \hat{\xi}_\mathrm{L} \| \tau$, $\hat{\xi}_\mathrm{T} = 0$
(c)	$\mathbb{Z}_2^\tau \times \mathbb{Z}_2^{\hat{\tau}\circ\kappa}$	$\mathbb{Z}_2^\tau \times \mathbb{Z}_2^{\hat{\tau}\circ\kappa}$	3	$\exp(\hat{\xi})$	$\hat{\xi}_\mathrm{R} = \hat{\xi}_\mathrm{L} = 0$, $\hat{\xi}_\mathrm{T} \| \tau$

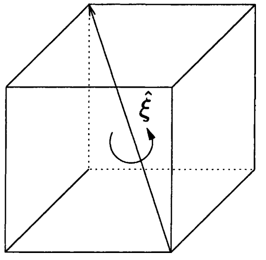

Fig. 6. Pulsating cube RPOs of Type 2.

Table 2
Symmetries of Type 1 relative periodic orbits bifurcating from the cubic oscillations [41]; τ and $\hat{\tau}$ are two different non-identity elements in $\mathbb{D}_2 = \mathrm{SO}(3)_{\mathrm{D}} \cap G_p$

	$\hat{G}_p$	$\hat{\Sigma}$	ℓ	$\hat{\sigma}$	$\hat{\xi}$
(a)	id	$\mathbb{Z}_3^{\alpha}$	1	$\alpha \exp(\hat{\xi})$	$\hat{\xi}_{\mathrm{R}} \| \hat{\xi}_{\mathrm{L}} \| \hat{\xi}_{\mathrm{T}} \| \alpha$
(b)	$\mathbb{Z}_2^{\kappa}$	$\mathbb{Z}_3^{\alpha} \times \mathbb{Z}_2^{\kappa}$	1	$\alpha \exp(\hat{\xi})$	$\hat{\xi}_{\mathrm{R}} \| \hat{\xi}_{\mathrm{L}} \| \alpha, \hat{\xi}_{\mathrm{T}} = 0$

family rotate around and translate along the rotation axis of α, i.e. the cross-diagonal of the cube, see Fig. 6.

This family contains a three-parameter family of relative periodic orbits near $\mathcal{P}$ with relative period close to 1, isotropy $\hat{G}_p = \mathbb{Z}_2^{\kappa}$ and drift symmetry close to α. To see this let $\hat{G}_p = \mathbb{Z}_2^{\kappa}$. Then $L = N(\hat{G}_p)/\hat{G}_p = \mathrm{O}(3)_{\mathrm{L}} \times \mathrm{O}(3)_{\mathrm{R}}$, $\mathbf{l} = \mathrm{Fix}_{\mathbf{g}}(\hat{G}_p) = \mathrm{so}(3)_{\mathrm{L}} \oplus \mathrm{so}(3)_{\mathrm{R}}$ and $\mathbf{l}_{\alpha} = \mathrm{so}(2)_{\mathrm{L}} \oplus \mathrm{so}(2)_{\mathrm{R}}$ so that α is minimal in L and Theorem 4.7 applies. The symmetry data of Type 2 family of relative periodic orbits and its subfamily are summarized in Table 2.

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